# 14.12 Game Theory Lecture Notes
# Introduction

Muhamet Yildiz

(Lecture 1)

Game Theory is a misnomer for Multiperson Decision Theory, analyzing the decision-making process when there are more than one decision-makers where each agent's payoff possibly depends on the actions taken by the other agents. Since an agent's preferences on his actions depend on which actions the other parties take, his action depends on his beliefs about what the others do. Of course, what the others do depends on their beliefs about what each agent does. In this way, a player's action, in principle, depends on the actions available to each agent, each agent's preferences on the outcomes, each player's beliefs about which actions are available to each player and how each player ranks the outcomes, and further his beliefs about each player's beliefs, ad infinitum.

Under perfect competition, there are also more than one (in fact, infinitely many) decision makers. Yet, their decisions are assumed to be decentralized. A consumer tries to choose the best consumption bundle that he can afford, given the prices – without paying attention what the other consumers do. In reality, the future prices are not known. Consumers' decisions depend on their expectations about the future prices. And the future prices depend on consumers' decisions today. Once again, even in perfectly competitive environments, a consumer's decisions are affected by their beliefs about what other consumers do – in an aggregate level.

When agents think through what the other players will do, taking what the other players think about them into account, they may find a clear way to play the game. Consider the following "game":

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| 1 \ 2 | L | m | R |
|---|---|---|---|
| T | $(1,1)$ | $(0,2)$ | $(2,1)$ |
| M | $(2,2)$ | $(1,1)$ | $(0,0)$ |
| B | $(1,0)$ | $(0,0)$ | $(-1,1)$ |

Here, Players 1 has strategies, T, M, B and Player 2 has strategies L, m, R. (They pick their strategies simultaneously.) The payoffs for players 1 and 2 are indicated by the numbers in parentheses, the first one for player 1 and the second one for player 2. For instance, if Player 1 plays T and Player 2 plays R, then Player 1 gets a payoff of 2 and Player 2 gets 1. Let's assume that each player knows that these are the strategies and the payoffs, each player knows that each player knows this, each player knows that each player knows that each player knows this,... ad infinitum.

Now, player 1 looks at his payoffs, and realizes that, no matter what the other player plays, it is better for him to play M rather than B. That is, if 2 plays L, M gives 2 and B gives 1; if 2 plays m, M gives 1, B gives 0; and if 2 plays R, M gives 0, B gives -1. Therefore, he realizes that he should not play B.[1] Now he compares T and M. He realizes that, if Player 2 plays L or m, M is better than T, but if she plays R, T is definitely better than M. Would Player 2 play R? What would she play? To find an answer to these questions, Player 1 looks at the game from Player 2's point of view. He realizes that, for Player 2, there is no strategy that is outright better than any other strategy. For instance, R is the best strategy if 1 plays B, but otherwise it is strictly worse than m. Would Player 2 think that Player 1 would play B? Well, she knows that Player 1 is trying to maximize his expected payoff, given by the first entries as everyone knows. She must then deduce that Player 1 will not play B. Therefore, Player 1 concludes, she will not play R (as it is worse than m in this case). Ruling out the possibility that Player 2 plays R, Player 1 looks at his payoffs, and sees that M is now better than T, no matter what. On the other side, Player 2 goes through similar reasoning, and concludes that 1 must play M, and therefore plays L.

This kind of reasoning does not always yield such a clear prediction. Imagine that you want to meet with a friend in one of two places, about which you both are indifferent. Unfortunately, you cannot communicate with each other until you meet. This situation

[1] After all, he cannot have any belief about what Player 2 plays that would lead him to play B when M is available.

is formalized in the following game, which is called *pure coordination game*:

| 1 \ 2 | Left | Right |
|---|---|---|
| Top | (1,1) | (0,0) |
| Bottom | (0,0) | (1,1) |

Here, Player 1 chooses between Top and Bottom rows, while Player 2 chooses between Left and Right columns. In each box, the first and the second numbers denote the von Neumann-Morgenstern utilities of players 1 and 2, respectively. Note that Player 1 prefers Top to Bottom if he knows that Player 2 plays Left; he prefers Bottom if he knows that Player 2 plays Right. He is indifferent if he thinks that the other player is likely to play either strategy with equal probabilities. Similarly, Player 2 prefers Left if she knows that player 1 plays Top. There is no clear prediction about the outcome of this game.

One may look for the *stable* outcomes (strategy profiles) in the sense that no player has incentive to deviate if he knows that the other players play the prescribed strategies. Here, Top-Left and Bottom-Right are such outcomes. But Bottom-Left and Top-Right are not stable in this sense. For instance, if Bottom-Left is known to be played, each player would like to deviate – as it is shown in the following figure:

| 1 \ 2 | Left | Right |
|---|---|---|
| Top | (1,1) | $\Leftarrow\Downarrow$(0,0) |
| Bottom | (0,0)$\Uparrow\Longrightarrow$ | (1,1) |

(Here, $\Uparrow$ means player 1 deviates to Top, etc.)

Unlike in this game, mostly players have different preferences on the outcomes, inducing *conflict*. In the following game, which is known as the *Battle of Sexes*, conflict and the need for coordination are present together.

| 1 \ 2 | Left | Right |
|---|---|---|
| Top | (2,1) | (0,0) |
| Bottom | (0,0) | (1,2) |

Here, once again players would like to coordinate on Top-Left or Bottom-Right, but now Player 1 prefers to coordinate on Top-Left, while Player 2 prefers to coordinate on Bottom-Right. The stable outcomes are again Top-Left and Bottom- Right.

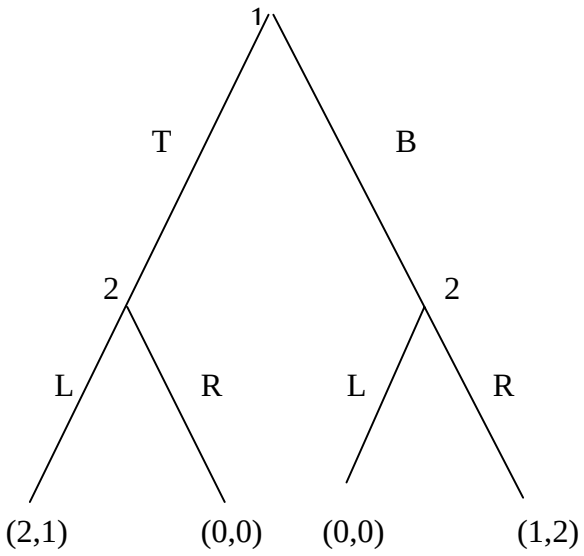



Figure 1:

Now, in the Battle of Sexes, imagine that Player 2 knows what Player 1 does when she takes her action. This can be formalized via the following tree:

Here, Player 1 chooses between Top and Bottom, then (knowing what Player 1 has chosen) Player 2 chooses between Left and Right. Clearly, now Player 2 would choose Left if Player 1 plays Top, and choose Right if Player 1 plays Bottom. Knowing this, Player 1 would play Top. Therefore, one can argue that the only reasonable outcome of this game is Top-Left. (This kind of reasoning is called *backward induction*.)

When Player 2 is to check what the other player does, he gets only 1, while Player 1 gets 2. (In the previous game, two outcomes were stable, in which Player 2 would get 1 or 2.) That is, Player 2 prefers that Player 1 has information about what Player 2 does, rather than she herself has information about what player 1 does. When it is common knowledge that a player has some information or not, the player may prefer not to have that information – a robust fact that we will see in various contexts.

**Exercise 1** *Clearly, this is generated by the fact that Player 1 knows that Player 2 will know what Player 1 does when she moves. Consider the situation that Player 1 thinks that Player 2 will know what Player 1 does only with probability* $\pi < 1$*, and this probability does not depend on what Player 1 does. What will happen in a "reasonable" equilibrium? [By the end of this course, hopefully, you will be able to formalize this*

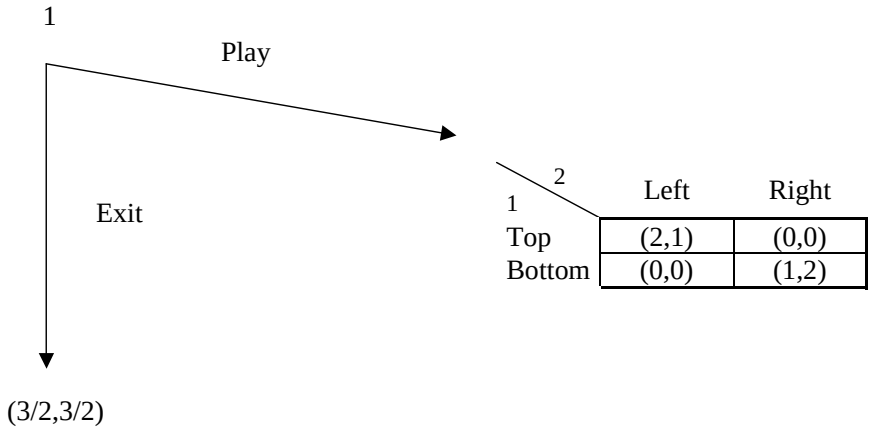
*situation, and compute the equilibria.]*

Another interpretation is that Player 1 can communicate to Player 2, who cannot communicate to player 1. This enables player 1 to commit to his actions, providing a strong position in the relation.

**Exercise 2** *Consider the following version of the last game: after knowing what Player 2 does, Player 1 gets a chance to change his action; then, the game ends. In other words, Player 1 chooses between Top and Bottom; knowing Player 1's choice, Player 2 chooses between Left and Right; knowing 2's choice, Player 1 decides whether to stay where he is or to change his position. What is the "reasonable" outcome? What would happen if changing his action would cost player 1 c utiles?*

Imagine that, before playing the Battle of Sexes, Player 1 has the option of exiting, in which case each player will get 3/2, or playing the Battle of Sexes. When asked to play, Player 2 will know that Player 1 chose to play the Battle of Sexes.

There are two "reasonable" equilibria (or stable outcomes). One is that Player 1 exits, thinking that, if he plays the Battle of Sexes, they will play the Bottom-Right equilibrium of the Battle of Sexes, yielding only 1 for player 1. The second one is that Player 1 chooses to Play the Battle of Sexes, and in the Battle of Sexes they play Top-Left equilibrium.




Some would argue that the first outcome is not really reasonable? Because, when asked to play, Player 2 will know that Player 1 has chosen to play the Battle of Sexes, forgoing the payoff of 3/2. She must therefore realize that Player 1 cannot possibly be

planning to play Bottom, which yields the payoff of 1 max. That is, when asked to play, Player 2 should understand that Player 1 is planning to play Top, and thus she should play Left. Anticipating this, Player 1 should choose to play the Battle of Sexes game, in which they play Top-Left. Therefore, the second outcome is the only reasonable one. (This kind of reasoning is called *Forward Induction*.)

Here are some more examples of games:

1. Prisoners' Dilemma:

| 1 \ 2 | Confess | Not Confess |
|---|---|---|
| Confess | (-1, -1) | (1, -10) |
| Not Confess | (-10, 1) | (0, 0) |

   This is a well known game that most of you know. [It is also discussed in Gibbons.] In this game no matter what the other player does, each player would like to confess, yielding (-1,-1), which is dominated by (0,0).

2. Hawk-Dove game

| 1 \ 2 | Hawk | Dove |
|---|---|---|
| Hawk | $\left(\frac{V-C}{2}, \frac{V-C}{2}\right)$ | $(V, 0)$ |
| Dove | $(0,V)$ | $(\frac{V}{2},\frac{V}{2})$ |

   This is a generic biological game, but is also quite similar to many games in economics and political science. $V$ is the value of a resource that one of the players will enjoy. If they shared the resource, their values are $V/2$. Hawk stands for a "tough" strategy, whereby the player does not give up the resource. However, if the other player is also playing hawk, they end up fighting, and incur the cost $C/2$ each. On the other hand, a Hawk player gets the whole resource for itself when playing a Dove. When $V > C$, we have a Prisoners' Dilemma game, where we would observe fight.

   When we have $V < C$, so that fighting is costly, this game is similar to another well-known game, inspired by the movie Rebel Without a Cause, named "Chicken", where two players driving towards a cliff have to decide whether to stop or continue. The one who stops first loses face, but may save his life. More generally, a class of games called "wars of attrition" are used to model this type of situations. In

this case, a player would like to play Hawk if his opponent plays Dove, and play Dove if his opponent plays Hawk.

# 14.12 Game Theory Lecture Notes
# Theory of Choice

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(Lecture 2)

## 1 The basic theory of choice

We consider a set $X$ of alternatives. Alternatives are mutually exclusive in the sense that one cannot choose two distinct alternatives at the same time. We also take the set of feasible alternatives exhaustive so that a player's choices will always be defined. Note that this is a matter of modeling. For instance, if we have options Coffee and Tea, we define alternatives as $C$ = Coffee but no Tea, $T$ = Tea but no Coffee, $CT$ = Coffee and Tea, and $NT$ = no Coffee and no Tea.

Take a relation $\succeq$ on $X$. Note that a relation on $X$ is a subset of $X \times X$. A relation $\succeq$ is said to be *complete* if and only if, given any $x, y \in X$, either $x \succeq y$ or $y \succeq x$. A relation $\succeq$ is said to be *transitive* if and only if, given any $x, y, z \in X$,

$$[x \succeq y \text{ and } y \succeq z] \Rightarrow x \succeq z.$$

A relation is a *preference relation* if and only if it is complete and transitive. Given any preference relation $\succeq$, we can define strict preference $\succ$ by

$$x \succ y \iff [x \succeq y \text{ and } y \not\succeq x],$$

and the indifference $\sim$ by

$$x \sim y \iff [x \succeq y \text{ and } y \succeq x].$$

A preference relation can be *represented* by a utility function $u : X \to \mathbb{R}$ in the following sense:

$$x \succeq y \Longleftrightarrow u(x) \geq u(y) \qquad \forall x, y \in X.$$

The following theorem states further that a relation needs to be a preference relation in order to be represented by a utility function.

**Theorem 1** *Let $X$ be finite. A relation can be presented by a utility function if and only if it is complete and transitive. Moreover, if $u : X \to \mathbb{R}$ represents $\succeq$, and if $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then $f \circ u$ also represents $\succeq$.*

By the last statement, we call such utility functions ordinal.

In order to use this ordinal theory of choice, we should know the agent's preferences on the alternatives. As we have seen in the previous lecture, in game theory, a player chooses between his strategies, and his preferences on his strategies depend on the strategies played by the other players. Typically, a player does not know which strategies the other players play. Therefore, we need a theory of decision-making under uncertainty.

# 2 Decision-making under uncertainty

We consider a finite set $Z$ of prizes, and the set $P$ of all probability distributions $p : Z \to [0, 1]$ on $Z$, where $\sum_{z \in Z} p(z) = 1$. We call these probability distributions lotteries. A lottery can be depicted by a tree. For example, in Figure 1, Lottery 1 depicts a situation in which if head the player gets \$10, and if tail, he gets \$0.

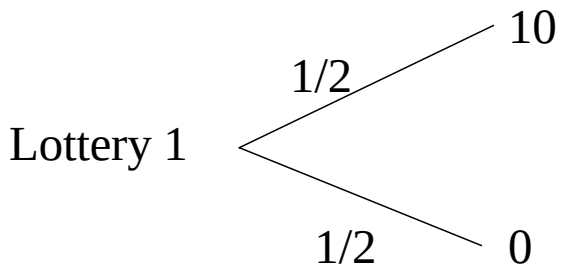



Figure 1:

Unlike the situation we just described, in game theory and more broadly when agents make their decision under uncertainty, we do not have the lotteries as in casinos where the probabilities are generated by some machines or given. Fortunately, it has been shown by Savage (1954) under certain conditions that a player's beliefs can be represented by

a (unique) probability distribution. Using these probabilities, we can represent our acts by lotteries.

We would like to have a theory that constructs a player's preferences on the lotteries from his preferences on the prizes. There are many of them. The most well-known—and the most canonical and the most useful—one is the theory of expected utility maximization by Von Neumann and Morgenstern. A preference relation $\succeq$ on $P$ is said to be represented by a von Neumann-Morgenstern utility function $u : Z \to \mathbb{R}$ if and only if

$$p \succeq q \iff U(p) \equiv \sum_{z \in Z} u(z)p(z) \geq \sum_{z \in Z} u(z)q(z) \equiv U(q) \tag{1}$$

for each $p, q \in P$. Note that $U : P \to \mathbb{R}$ represents $\succeq$ in ordinal sense. That is, the agent acts as if he wants to maximize the expected value of $u$. For instance, the expected utility of Lottery 1 for our agent is $E(u(\text{Lottery 1})) = \frac{1}{2}u(10) + \frac{1}{2}u(0)$.[1]

The necessary and sufficient conditions for a representation as in (1) are as follows:

**Axiom 1** *$\succeq$ is complete and transitive.*

This is necessary by Theorem 1, for $U$ represents $\succeq$ in ordinal sense. The second condition is called *independence* axiom, stating that a player's preference between two lotteries $p$ and $q$ does not change if we toss a coin and give him a fixed lottery $r$ if "tail" comes up.

**Axiom 2** *For any $p, q, r \in P$, and any $a \in (0, 1]$, $ap + (1-a)r \succ aq + (1-a)r \iff p \succ q$.*

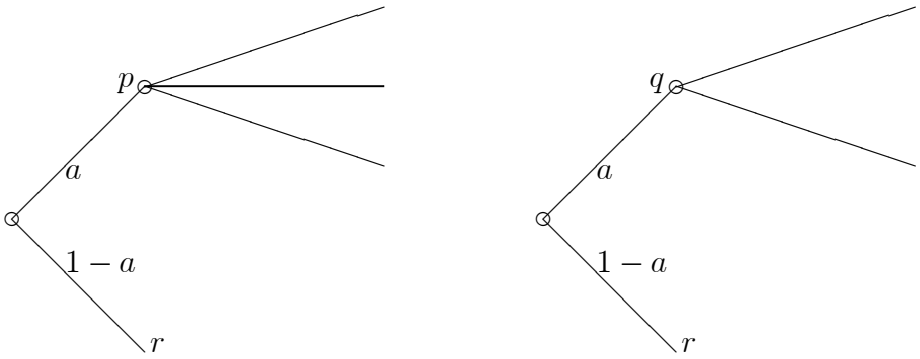
Let $p$ and $q$ be the lotteries depicted in Figure 2. Then, the lotteries $ap + (1-a)r$ and $aq + (1-a)r$ can be depicted as in Figure 3, where we toss a coin between a fixed lottery $r$ and our lotteries $p$ and $q$. Axiom 2 stipulates that the agent would not change his mind after the coin toss. Therefore, our axiom can be taken as an axiom of "dynamic consistency" in this sense.

The third condition is purely technical, and called *continuity* axiom. It states that there are no "infinitely good" or "infinitely bad" prizes.

**Axiom 3** *For any $p, q, r \in P$, if $p \succ r$, then there exist $a, b \in (0, 1)$ such that $ap + (1-a)r \succ q \succ bp + (1-r)r$.*

[1] If $Z$ were a continuum, like $\mathbb{R}$, we would compute the expected utility of $p$ by $\int u(z)p(z)dz$.

Figure 2: Two lotteries




$ap + (1 - a)r$ $aq + (1 - a)r$

Figure 3: Two compound lotteries

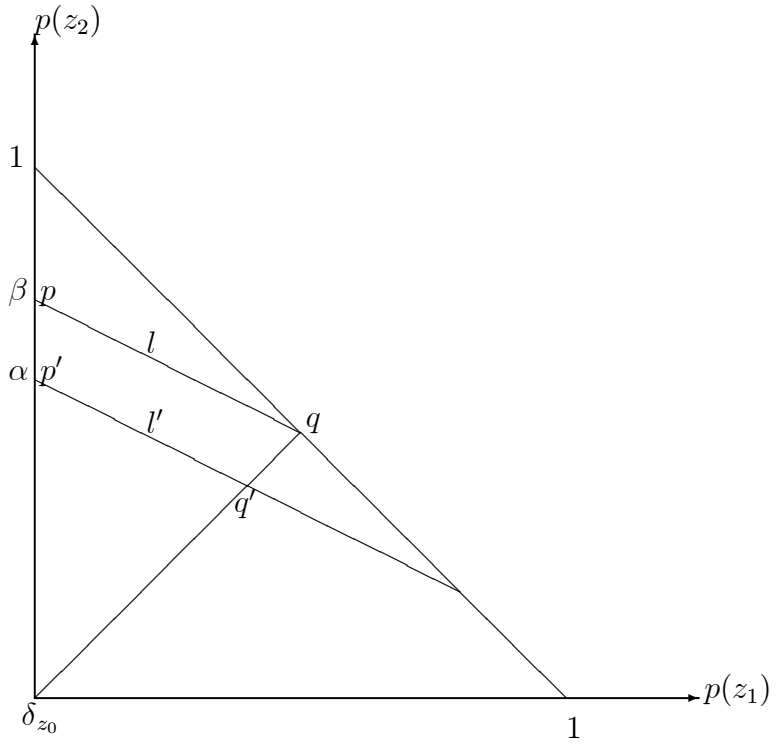



Figure 4: Indifference curves on the space of lotteries

Axioms 2 and 3 imply that, given any $p, q, r \in P$ and any $a \in [0,1]$,

$$\text{if } p \sim q, \text{ then } ap + (1-a)\, r \sim aq + (1-a)r. \tag{2}$$

This has two implications:

1. The indifference curves on the lotteries are straight lines.

2. The indifference curves, which are straight lines, are parallel to each other.

To illustrate these facts, consider three prizes $z_0, z_1$, and $z_2$, where $z_2 \succ z_1 \succ z_0$. A lottery $p$ can be depicted on a plane by taking $p(z_1)$ as the first coordinate (on the horizontal axis), and $p(z_2)$ as the second coordinate (on the vertical axis). $p(z_0)$ is $1 - p(z_1) - p(z_2)$. [See Figure 4 for the illustration.] Given any two lotteries $p$ and $q$, the convex combinations $ap + (1-a)\, q$ with $a \in [0,1]$ form the line segment connecting $p$ to $q$. Now, taking $r = q$, we can deduce from (2) that, if $p \sim q$, then

$ap + (1-a)\,q \sim aq + (1-a)q = q$ for each $a \in [0,1]$. That this, the line segment connecting $p$ to $q$ is an indifference curve. Moreover, if the lines $l$ and $l'$ are parallel, then $\alpha/\beta = |q'|\,/\,|q|$, where $|q|$ and $|q'|$ are the distances of $q$ and $q'$ to the origin, respectively. Hence, taking $a = \alpha/\beta$, we compute that $p' = ap + (1-a)\,\delta_{z_0}$ and $q' = aq + (1-a)\,\delta_{z_0}$, where $\delta_{z_0}$ is the lottery at the origin, and gives $z_0$ with probability 1. Therefore, by (2), if $l$ is an indifference curve, $l'$ is also an indifference curve, showing that the indifference curves are parallel.

Line $l$ can be defined by equation $u_1 p(z_1) + u_2 p(z_2) = c$ for some $u_1, u_2, c \in \mathbb{R}$. Since $l'$ is parallel to $l$, then $l'$ can also be defined by equation $u_1 p(z_1) + u_2 p(z_2) = c'$ for some $c'$. Since the indifference curves are defined by equality $u_1 p(z_1) + u_2 p(z_2) = c$ for various values of $c$, the preferences are represented by

$$\begin{aligned} U(p) &= 0 + u_1 p(z_1) + u_2 p(z_2) \\ &\equiv u(z_0)p(z_0) + u(z_1)p(z_1) + u(z_2)p(z_2), \end{aligned}$$

where

$$\begin{aligned} u(z_0) &= 0, \\ u(z_1) &= u_1, \\ u(z_2) &= u_2, \end{aligned}$$

giving the desired representation.

This is true in general, as stated in the next theorem:

**Theorem 2** *A relation $\succeq$ on $P$ can be represented by a von Neumann-Morgenstern utility function $u : Z \to R$ as in (1) if and only if $\succeq$ satisfies Axioms 1-3. Moreover, $u$ and $\tilde{u}$ represent the same preference relation if and only if $\tilde{u} = au + b$ for some $a > 0$ and $b \in \mathbb{R}$.*

By the last statement in our theorem, this representation is "unique up to affine transformations". That is, an agent's preferences do not change when we change his von Neumann-Morgenstern (VNM) utility function by multiplying it with a positive number, or adding a constant to it; but they do change when we transform it through a non-linear transformation. In this sense, this representation is "cardinal". Recall that, in ordinal representation, the preferences wouldn't change even if the transformation

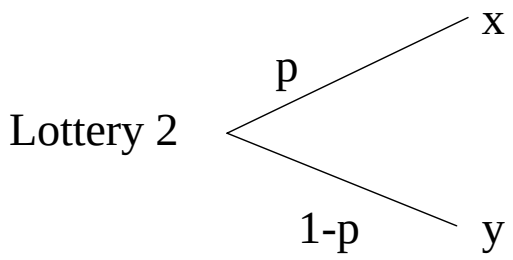
were non-linear, so long as it was increasing. For instance, under certainty, $v = \sqrt{u}$ and $u$ would represent the same preference relation, while (when there is uncertainty) the VNM utility function $v = \sqrt{u}$ represents a very different set of preferences on the lotteries than those are represented by $u$. Because, in cardinal representation, the curvature of the function also matters, measuring the agent's attitudes towards risk.

# 3 Attitudes Towards Risk

Suppose individual A has utility function $u_A$. How do we determine whether he dislikes risk or not?

The answer lies in the cardinality of the function $u$.

Let us first define a *fair* gamble, as a lottery that has expected value equal to 0. For instance, lottery 2 below is a fair gamble if and only if $px + (1-p)y = 0$.




We define an agent as *Risk-Neutral* if and only if he is indifferent between accepting and rejecting all fair gambles. Thus, an agent with utility function $u$ is risk neutral if and only if

$$E(u(\text{lottery 2})) = pu(x) + (1-p)u(y) = u(0)$$

for all $p$, $x$, and $y$.

This can only be true for all $p$, $x$, and $y$ if and only if the agent is maximizing the expected value, that is, $u(x) = ax + b$. Therefore, we need the utility function to be linear.

Therefore, an agent is risk-neutral if and only if he has a linear Von-Neumann-Morgenstern utility function.

An agent is *strictly risk-averse* if and only if he rejects *all* fair gambles:

$$E(u(\text{lottery 2})) < u(0)$$
$$pu(x) + (1-p)u(y) < u(px + (1-p)y) \equiv u(0)$$

Now, recall that a function $g(\cdot)$ is strictly concave if and only if we have

$$g(\lambda x + (1-\lambda)y) > \lambda g(x) + (1-\lambda)g(y)$$

for all $\lambda \in (0,1)$. Therefore, strict risk-aversion is equivalent to having a strictly concave utility function. We will call an agent *risk-averse* iff he has a *concave* utility function, i.e., $u(\lambda x + (1-\lambda)y) > \lambda u(x) + (1-\lambda)u(y)$ for each $x$, $y$, and $\lambda$.

Similarly, an agent is said to be (strictly) risk seeking iff he has a (strictly) convex utility function.

Consider Figure 5. The cord AB is the utility difference that this risk-averse agent would lose by taking the gamble that gives $W_1$ with probability $p$ and $W_2$ with probability $1-p$. BC is the maximum amount that she would pay in order to avoid to take the gamble. Suppose $W_2$ is her wealth level and $W_2 - W_1$ is the value of her house and p is the probability that the house burns down. Thus in the absence of fire insurance this individual will have utility given by $EU(\text{gamble})$, which is lower than the utility of the expected value of the gamble.

## 3.1 Risk sharing

Consider an agent with utility function $u : x \mapsto \sqrt{x}$. He has a (risky) asset that gives \$100 with probability 1/2 and gives \$0 with probability 1/2. The expected utility of our agent from this asset is $EU_0 = \frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{100} = 5$. Now consider another agent who is identical to our agent, in the sense that he has the same utility function and an asset that pays \$100 with probability 1/2 and gives \$0 with probability 1/2. We assume throughout that what an asset pays is statistically independent from what the other asset pays. Imagine that our agents form a mutual fund by pooling their assets, each agent owning half of the mutual fund. This mutual fund gives \$200 the probability 1/4 (when both assets yield high dividends), \$100 with probability 1/2 (when only one on the assets gives high dividend), and gives \$0 with probability 1/4 (when both assets yield low dividends). Thus, each agent's share in the mutual fund yields \$100 with probability

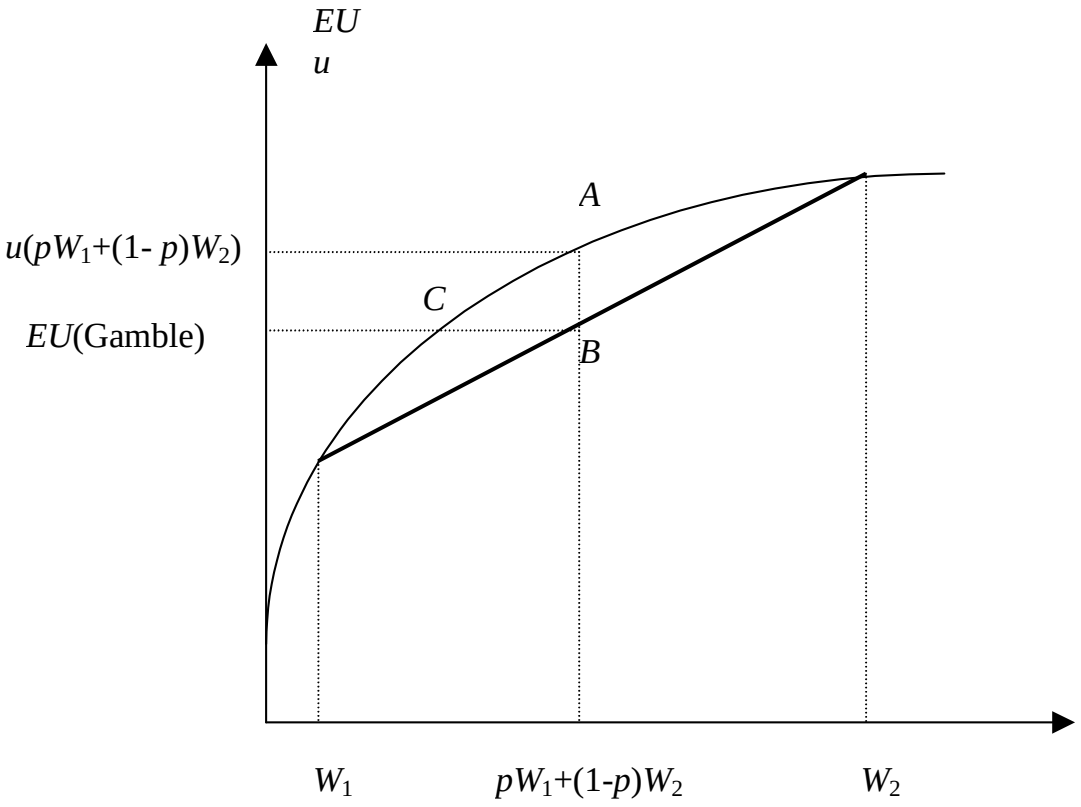



Figure 5:

1/4, \$50 with probability 1/2, and \$0 with probability 1/4. Therefore, his expected utility from the share in this mutual fund is $EU_S = \frac{1}{4}\sqrt{100} + \frac{1}{2}\sqrt{50} + \frac{1}{4}\sqrt{0} = 6.0355$. This is clearly larger than his expected utility from his own asset. Therefore, our agents gain from sharing the risk in their assets.

## 3.2 Insurance

Imagine a world where in addition to one of the agents above (with utility function $u : x \mapsto \sqrt{x}$ and a risky asset that gives \$100 with probability 1/2 and gives \$0 with probability 1/2), we have a risk-neutral agent with lots of money. We call this new agent the insurance company. The insurance company can insure the agent's asset, by giving him \$100 if his asset happens to yield \$0. How much premium, $P$, our risk averse agent would be willing to pay to get this insurance? [A premium is an amount that is to be paid to insurance company regardless of the outcome.]

If the risk-averse agent pays premium $P$ and buys the insurance his wealth will be $\$100 - P$ for sure. If he does not, then his wealth will be \$100 with probability 1/2 and \$0 with probability 1/2. Therefore, he will be willing to pay $P$ in order to get the insurance iff

$$u\left(100 - P\right) \geq \frac{1}{2}u\left(0\right) + \frac{1}{2}u\left(100\right)$$

i.e., iff

$$\sqrt{100 - P} \geq \frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{100}$$

iff

$$P \leq 100 - 25 = 75.$$

On the other hand, if the insurance company sells the insurance for premium $P$, it will get $P$ for sure and pay \$100 with probability 1/2. Therefore it is willing to take the deal iff

$$P \geq \frac{1}{2}100 = 50.$$

Therefore, both parties would gain, if the insurance company insures the asset for a premium $P \in (50, 75)$, a deal both parties are willing to accept.

**Exercise 3** *Now consider the case that we have two identical risk-averse agents as above, and the insurance company. Insurance company is to charge the same premium*

*P for each agent, and the risk-averse agents have an option of forming a mutual fund. What is the range of premiums that are acceptable to all parties?*

# 14.12 Game Theory Lecture Notes*
# Lectures 3-6

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We will formally define the games and some solution concepts, such as Nash Equilibrium, and discuss the assumptions behind these solution concepts.

In order to analyze a game, we need to know

- who the players are,
- which actions are available to them,
- how much each player values each outcome,
- what each player knows.

Notice that we need to specify not only what each player knows about external parameters, such as the payoffs, but also about what they know about the other players' knowledge and beliefs about these parameters, etc. In the first half of this course, we will confine ourselves to the games of complete information, where everything that is known by a player is common knowledge.[1] (We say that X is common knowledge if

*These notes are somewhat incomplete — they do not include some of the topics covered in the class.

†Some parts of these notes are based on the notes by Professor Daron Acemoglu, who taught this course before.

[1]Knowledge is defined as an operator on the propositions satisfying the following properties:

1. if I know X, X must be true;
2. if I know X, I know that I know X;
3. if I don't know X, I know that I don't know X;
4. if I know something, I know all its logical implications.

everyone knows X, and everyone knows that everyone knows X, and everyone knows that everyone knows that everyone knows X, ad infinitum.) In the second half, we will relax this assumption and allow player to have asymmetric information, focusing on informational issues.

# 1 Representations of games

The games can be represented in two forms:

1. The normal (strategic) form,
2. The extensive form.

## 1.1 Normal form

**Definition 1** *(Normal form) An n-player game is any list* $G = (S_1, \ldots, S_n; u_1, \ldots, u_n)$*, where, for each* $i \in N = \{1, \ldots, n\}$*,* $S_i$ *is the set of all strategies that are available to player* $i$*, and* $u_i : S_1 \times \ldots \times S_n \to \mathbb{R}$ *is player* $i$*'s von Neumann-Morgenstern utility function.*

Notice that a player's utility depends not only on his own strategy but also on the strategies played by other players. Moreover, each player $i$ tries to maximize the expected value of $u_i$ (where the expected values are computed with respect to his own beliefs); in other words, $u_i$ is a von Neumann-Morgenstern utility function. We will say that player $i$ is rational iff he tries to maximize the expected value of $u_i$ (given his beliefs).[2]

It is also assumed that it is common knowledge that the players are $N = \{1, \ldots, n\}$, that the set of strategies available to each player $i$ is $S_i$, and that each $i$ tries to maximize expected value of $u_i$ given his beliefs.

When there are only two players, we can represent the (normal form) game by a bimatrix (i.e., by two matrices):

| 1\2 | left | right |
|---|---|---|
| up | 0,2 | 1,1 |
| down | 4,1 | 3,2 |

[2] We have also made another very strong "rationality" assumption in defining knowledge, by assuming that, if I know something, then I know all its logical consequences.

Here, Player 1 has strategies up and down, and 2 has the strategies left and right. In each box the first number is 1's payoff and the second one is 2's (e.g., $u_1$ (up,left) = 0, $u_2$ (up,left) = 2.)

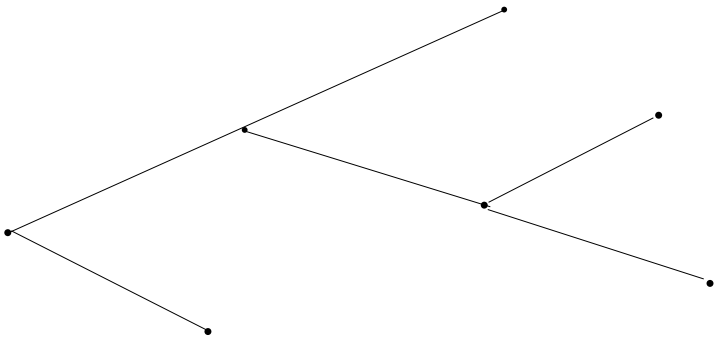
## 1.2 Extensive form

The extensive form contains all the information about a game, by defining who moves when, what each player knows when he moves, what moves are available to him, and where each move leads to, etc., (whereas the normal form is more of a 'summary' representation). We first introduce some formalisms.

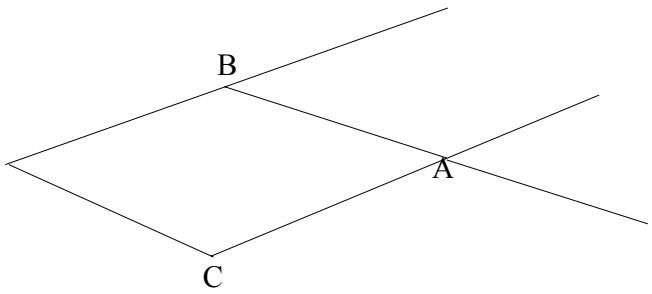
**Definition 2** *A* tree *is a set of nodes and directed edges connecting these nodes such that*

1. *there is an initial node, for which there is no incoming edge;*
2. *for every other node, there is one incoming edge;*
3. *for any two nodes, there is a unique path that connect these two nodes.*

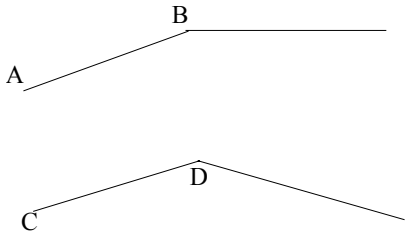
Imagine the branches of a tree arising from the trunk. For example,



is a tree. On the other hand,




is not a tree because there are two alternative paths through which point A can be reached (via B and via C).




is not a tree either since A and B are not connected to C and D.

**Definition 3** ***(Extensive form)*** *A* Game *consists of a set of players, a tree, an allocation of each node of the tree (except the end nodes) to a player, an informational partition, and payoffs for each player at each end node.*

The set of players will include the agents taking part in the game. However, in many games there is room for chance, e.g. the throw of dice in backgammon or the card draws in poker. More broadly, we need to consider "chance" whenever there is uncertainty about some relevant fact. To represent these possibilities we introduce a fictional player: Nature. There is no payoff for Nature at end nodes, and every time a node is allocated to Nature, a probability distribution over the branches that follow needs to be specified, e.g., Tail with probability of 1/2 and Head with probability of 1/2.

An *information set* is a collection of points (nodes) $\{n_1, \ldots, n_k\}$ such that

1. the same player $i$ is to move at each of these nodes;

2. the same moves are available at each of these nodes.

Here the player $i$, who is to move at the information set, is assumed to be unable to distinguish between the points in the information set, but able to distinguish between the points outside the information set from those in it. For instance, consider the game in Figure 1. Here, Player 2 knows that Player 1 has taken action T or B and not action X; but Player 2 cannot know for sure whether 1 has taken T or B. The same game is depicted in Figure 2 slightly differently.

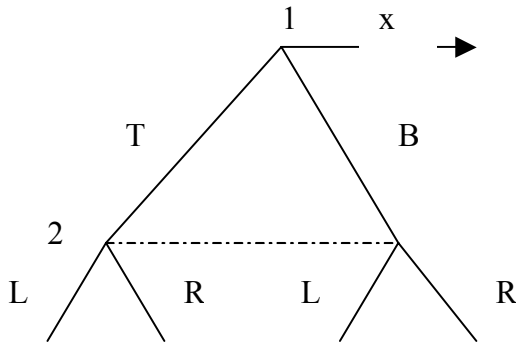



Figure 1:

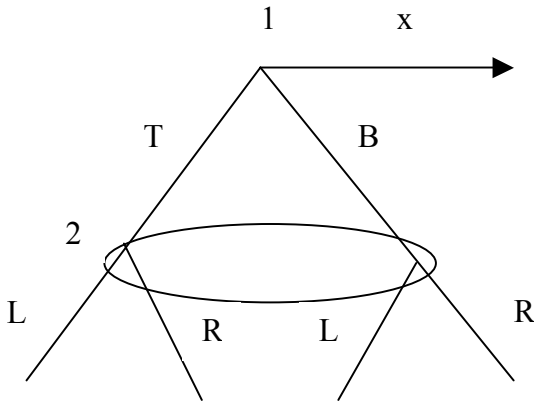



Figure 2:

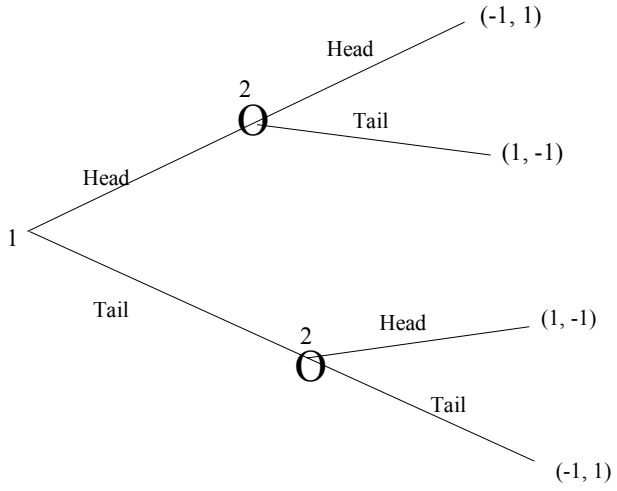
An *information partition* is an allocation of each node of the tree (except the starting and end-nodes) to an information set.

**To sum up:** at any node, we know: which player is to move, which moves are available to the player, and which information set contains the node, summarizing the player's information at the node. Of course, if two nodes are in the same information set, the available moves in these nodes must be the same, for otherwise the player could distinguish the nodes by the available choices. Again, all these are assumed to be common knowledge. For instance, in the game in Figure 1, player 1 knows that, if player 1 takes X, player 2 will know this, but if he takes T or B, player 2 will not know which of these two actions has been taken. (She will know that either T or B will have been taken.)

**Definition 4** A strategy *of a player* is a complete contingent-plan *determining which action he will take at each information set he is to move (including the information sets that will not be reached according to this strategy).*

For certain purposes it might suffice to look at the reduced-form strategies. A reduced form strategy is defined as an incomplete contingent plan that determines which action the agent will take at each information set he is to move and that has not been precluded by this plan. But for many other purposes we need to look at all the strategies. Let us now consider some examples:

**Game 1: Matching Pennies with Perfect Information**




The tree consists of 7 nodes. The first one is allocated to player 1, and the next two to player 2. The four end-nodes have payoffs attached to them. Since there are

two players, payoff vectors have two elements. The first number is the payoff of player 1 and the second is the payoff of player 2. These payoffs are von Neumann-Morgenstern utilities so that we can take expectations over them and calculate expected utilities.

The informational partition is very simple; all nodes are in their own information set. In other words, all information sets are singletons (have only 1 element). This implies that there is no uncertainty regarding the previous play (history) in the game. At this point recall that in a tree, each node is reached through a unique path. Therefore, if all information sets are singletons, a player can construct the history of the game perfectly. For instance in this game, player 2 knows whether player 1 chose Head or Tail. And player 1 knows that when he plays Head or Tail, Player 2 will know what player 1 has played. (Games in which all information sets are singletons are called *games of perfect information*.)

In this game, the set of strategies for player 1 is {Head, Tail}. A strategy of player 2 determines what to do depending on what player 1 does. So, his strategies are:

HH = Head if 1 plays Head, and Head if 1 plays Tail;

HT = Head if 1 plays Head, and Tail if 1 plays Tail;

TH = Tail if 1 plays Head, and Head if 1 plays Tail;

TT = Tail if 1 plays Head, and Tail if 1 plays Tail.

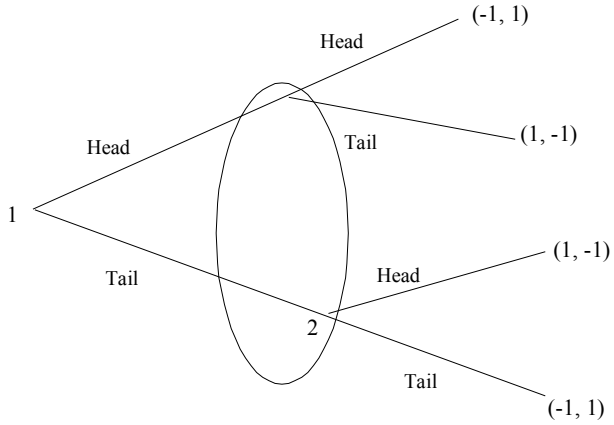
What are the payoffs generated by each strategy pair? If player 1 plays Head and 2 plays HH, then the outcome is [1 chooses Head and 2 chooses Head] and thus the payoffs are (-1,1). If player 1 plays Head and 2 plays HT, the outcome is the same, hence the payoffs are (-1,1). If 1 plays Tail and 2 plays HT, then the outcome is [1 chooses Tail and 2 chooses Tail] and thus the payoffs are once again (-1,1). However, if 1 plays Tail and 2 plays HH, then the outcome is [1 chooses Tail and 2 chooses Head] and thus the payoffs are (1,-1). One can compute the payoffs for the other strategy pairs similarly.

Therefore, the normal or the strategic form game corresponding to this game is

| | HH | HT | TH | TT |
|---|---|---|---|---|
| Head | -1,1 | -1,1 | 1,-1 | 1,-1 |
| Tail | 1,-1 | -1,1 | 1,-1 | -1,1 |

Information sets are very important! To see this, consider the following game.

**Game 2: Matching Pennies with Imperfect Information**

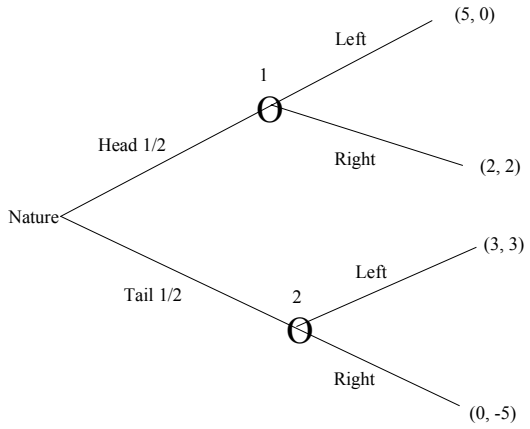



Games 1 and 2 appear very similar but in fact they correspond to two very different situations. In Game 2, when she moves, player 2 does not know whether 1 chose Head or Tail. This is a game of imperfect information (That is, some of the information sets contain more than one node.)

The strategies for player 1 are again Head and Tail. This time player 2 has also only two strategies: Head and Tail (as he does not know what 1 has played). The normal form representation for this game will be:

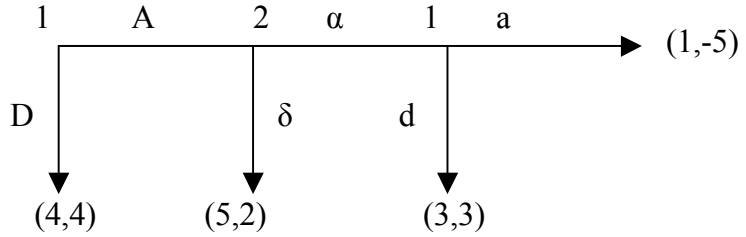
| 1\2 | Head | Tail |
|---|---|---|
| Head | -1,1 | 1,-1 |
| Tail | 1,-1 | -1,1 |

**Game 3: A Game with Nature:**



Here, we toss a fair coin, where the probability of Head is 1/2. If Head comes up, Player 1 chooses between Left and Right; if Tail comes up, Player 2 chooses between Left and Right.

**Exercise 5** *What is the normal-form representation for the following game:*




*Can you find another extensive-form game that has the same normal-form representation?*

[Hint: For each extensive-form game, there is only one normal-form representation (up to a renaming of the strategies), but a normal-form game typically has more than one extensive-form representation.]

In many cases a player may not be able to guess exactly which strategies the other players play. In order to cover these situations we introduce the mixed strategies:

**Definition 6** *A* mixed strategy *of a player is a probability distribution over the set of his strategies.*

If player $i$ has strategies $S_i = \{s_{i1}, s_{i2}, \ldots, s_{ik}\}$, then a mixed strategy $\sigma_i$ for player $i$ is a function on $S_i$ such that $0 \leq \sigma_i(s_{ij}) \leq 1$ and $\sigma_i(s_{i1}) + \sigma_i(s_{i2}) + \cdots + \sigma_i(s_{ik}) = 1$. Here $\sigma_i$ represents other players' beliefs about which strategy $i$ would play.

# 2 How to play?

We will now describe the most common "solution concepts" for normal-form games. We will first describe the concept of "dominant strategy equilibrium," which is implied by the rationality of the players. We then discuss "rationalizability" which corresponds to the common knowledge of rationality, and finally we discuss the Nash Equilibrium, which is related to the mutual knowledge of players' conjectures about the other players' actions.

## 2.1 Dominant-strategy equilibrium

Let us use the notation $s_{-i}$ to mean the list of strategies $s_j$ played by all the players $j$ other than $i$, i.e.,

$$s_{-i} = (s_1, ...s_{i-1}, s_{i+1}, ...s_n).$$

**Definition 7** *A strategy* $s_i^*$ strictly dominates $s_i$ *if and only if*

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i}.$$

That is, no matter what the other players play, playing $s_i^*$ is strictly better than playing $s_i$ for player $i$. In that case, if $i$ is rational, he would never play the strictly dominated strategy $s_i$.[3]

A mixed strategy $\sigma_i$ dominates a strategy $s_i$ in a similar way: $\sigma_i$ *strictly dominates* $s_i$ if and only if

$$\sigma_i(s_{i1})u_i(s_{i1}, s_{-i}) + \sigma_i(s_{i2})u_i(s_{i2}, s_{-i}) + \cdots \sigma_i(s_{ik})u_i(s_{ik}, s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i}.$$

A rational player $i$ will never play a strategy $s_i$ iff $s_i$ is dominated by a (mixed or pure) strategy.

Similarly, we can define weak dominance.

**Definition 8** *A strategy* $s_i^*$ weakly dominates $s_i$ *if and only if*

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i}$$

*and*

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i})$$

*for some* $s_{-i} \in S_{-i}$.

That is, no matter what the other players play, playing $s_i^*$ is at least as good as playing $s_i$, and there are some contingencies in which playing $s_i^*$ is strictly better than $s_i$. In that case, if rational, $i$ would play $s_i$ only if he believes that these contingencies will never occur. If he is cautious in the sense that he assigns some positive probability for each contingency, he will not play $s_i$.

[3] That is, there is no belief under which he would play $s_i$. Can you prove this?

**Definition 9** *A strategy $s_i^d$ is a* (weakly) dominant strategy *for player $i$ if and only if $s_i^d$ weakly dominates all the other strategies of player $i$. A strategy $s_i^d$ is a* strictly dominant strategy *for player $i$ if and only if $s_i^d$ strictly dominates all the other strategies of player $i$.*

If $i$ is rational, and has a strictly dominant strategy $s_i^d$, then he will not play any other strategy. If he has a weakly dominant strategy and cautious, then he will not play other strategies.

**Example:**

| 1\2 | work hard | shirk |
|---|---|---|
| hire | 2,2 | 1,3 |
| don't hire | 0,0 | 0,0 |

In this game, player 1 (firm) has a strictly dominant strategy which is to "hire." Player 2 has only a weakly dominated strategy. If players are rational, and in addition player 2 is cautious, then we expect player 1 to "hire", and player 2 to "shirk":[4]

| 1\2 | work hard | shirk |
|---|---|---|
| hire | 2,2 $\Longrightarrow$ | 1,3 |
| don't hire | 0,0 $\Uparrow$ | 0,0 $\Uparrow$ |

**Definition 10** *A strategy profile $s^d = (s_1^d, s_2^d, ....s_N^d)$ is a* dominant strategy equilibrium, *if and only if $s_i^d$ is a dominant strategy for each player $i$.*

As an example consider the Prisoner's Dilemma.

| 1\2 | confess | don't confess |
|---|---|---|
| confess | -5,-5 | 0,-6 |
| don't confess | -6,0 | -1,-1 |

"Confess" is a strictly dominant strategy for both players, therefore ("confess", "confess") is a dominant strategy equilibrium.

| 1\2 | confess | don't confess |
|---|---|---|
| confess | -5,-5 | $\Longleftarrow$ 0,-6 |
| don't confess | -6,0 $\Uparrow$ | $\Longleftarrow$-1,-1 $\Uparrow$ |

[4]This is the only outcome, provided that each player is rational and player 2 knows that player 1 is rational. Can you show this?

**Example: (second-price auction)** We have an object to be sold through an auction. There are two buyers. The value of the object for any buyer $i$ is $v_i$, which is known by the buyer $i$. Each buyer $i$ submits a bid $b_i$ in a sealed envelope, simultaneously. Then, we open the envelopes;

the agent $i^*$ who submits the highest bid

$$b_{i^*} = \max\{b_1, b_2\}$$

gets the object and pays the second highest bid (which is $b_j$ with $j \neq i^*$). (If two or more buyers submit the highest bid, we select one of them by a coin toss.)

Formally the game is defined by the player set $N = \{1, 2\}$, the strategies $b_i$, and the payoffs

$$u_i(b_1, b_2) = \begin{cases} v_i - b_j & \text{if } b_i > b_j \\ (v_i - b_j)/2 & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

where $i \neq j$.

In this game, bidding his true valuation $v_i$ is a dominant strategy for each player $i$. To see this, consider the strategy of bidding some other value $b_i' \neq v_i$ for any $i$. We want to show that $b_i'$ is weakly dominated by bidding $v_i$. Consider the case $b_i' < v_i$. If the other player bids some $b_j < b_i'$, player $i$ would get $v_i - b_j$ under both strategies $b_i'$ and $v_i$. If the other player bids some $b_j \geq v_i$, player $i$ would get 0 under both strategies $b_i'$ and $v_i$. But if $b_j = b_i'$, bidding $v_i$ yields $v_i - b_j > 0$, while $b_i'$ yields only $(v_i - b_j)/2$. Likewise, if $b_i' < b_j < v_i$, bidding $v_i$ yields $v_i - b_j > 0$, while $b_i'$ yields only 0. Therefore, bidding $v_i$ dominates $b_i'$. The case $b_i' > v_i$ is similar, except for when $b_i' > b_j > v_i$, bidding $v_i$ yields 0, while $b_i'$ yields negative payoff $v_i - b_j < 0$. Therefore, bidding $v_i$ is dominant strategy for each player $i$.

**Exercise 11** *Extend this to the n-buyer case.*

When it exists, the dominant strategy equilibrium has an obvious attraction. In that case, the rationality of players implies that the dominant strategy equilibrium will be played. However, it does not exist in general. The following game, the Battle of the Sexes, is supposed to represent a timid first date (though there are other games from animal behavior that deserve this title much more). Both the man and the woman

want to be together rather than go alone. However, being timid, they do not make a firm date. Each is hoping to find the other either at the opera or the ballet. While the woman prefers the ballet, the man prefers the opera.

| Man\Woman | opera | ballet |
|---|---|---|
| opera | 1,4 | 0,0 |
| ballet | 0,0 | 4,1 |

Clearly, no player has a dominant strategy:

| Man\Woman | opera | ballet |
|---|---|---|
| opera | 1,4 | $\Leftarrow \Downarrow$ 0,0 |
| ballet | 0,0 $\Uparrow \Rightarrow$ | 4,1 |

## 2.2 Rationalizability or Iterative elimination of strictly dominated strategies

Consider the following Extended Prisoner's Dilemma game:

| 1\2 | confess | don't confess | run away |
|---|---|---|---|
| confess | -5,-5 | 0,-6 | -5,-10 |
| don't confess | -6,0 | -1,-1 | 0,-10 |
| run away | -10,-6 | -10,0 | -10,-10 |

In this game, no agent has any dominant strategy, but there exists a dominated strategy: "run away" is strictly dominated by "confess" (both for 1 and 2). Now consider 2's problem. She knows 1 is "rational," therefore she can predict that 1 will not choose "run away," thus she can eliminate "run away" and consider the smaller game

| 1\2 | confess | don't confess | run away |
|---|---|---|---|
| confess | -5,-5 | 0,-6 | -5,-10 |
| don't confess | -6,0 | -1,-1 | 0,-10 |

where we have eliminated "run away" because it was strictly dominated; the column player reasons that the row player would never choose it.

In this smaller game, 2 has a dominant strategy which is to "confess." That is, if 2 is rational and knows that 1 is rational, she will play "confess."

In the original game "don't confess" did better against "run away," thus "confess" was not a dominant strategy. However, player 1 playing "run away" cannot be *rationalized* because it is a dominated strategy. This leads to the *Elimination of Strictly Dominated Strategies*. What happens if we "Iteratively Eliminate Strictly Dominated" strategies? That is, we eliminate a strictly dominated strategy, and then look for another strictly dominated strategy in the reduced game. We stop when we can no longer find a strictly dominated strategy. Clearly, if it is common knowledge that players are rational, they will play only the strategies that survive this iteratively elimination of strictly dominated strategies. Therefore, we call such strategies *rationalizable*. **Caution**: we do eliminate the strategies that are dominated by some mixed strategies!

In the above example, the set of rationalizable strategies is once again "confess," "confess."

**At this point you should stop and apply this method to the Cournot duopoly!!** (See Gibbons.) Also, make sure that you can generate the rationality assumption at each elimination. For instance, in the game above, player 2 knows that player 1 is rational and hence he will not "run away;" and since she is also rational, she will play only "confess," for the "confess" is the only best response for any belief of player 2 that assigns 0 probability to that player 1 "runs away."

The problem is there may be too many rationalizable strategies. Consider the Matching Pannies game:

| 1\2 | Head | Tail |
|---|---|---|
| Head | -1,1 | 1,-1 |
| Tail | 1,-1 | -1,1 |

Here, every strategy is rationalizable. For example, if player 1 believes that player 2 will play Head, then he will play Tail, and if player 2 believes that player 1 will play Tail, then she will play Tail. Thus, the strategy-pair (Head,Tail) is rationalizable. But note that the beliefs of 1 and 2 are not congruent.

The set of rationalizable strategies is in general very large. In contrast, the concept of dominant strategy equilibrium is too restrictive: usually it does not exist.

The reason for existence of too many rationalizable strategies is that we do not restrict players' conjectures to be 'consistent' with what the others are actually doing. For instance, in the rationalizable strategy (Head, Tail), player 2 plays Tail by conjecturing that Player 1 will play Tail, while Player 1 actually plays Head. We consider another concept — Nash Equilibrium (henceforth NE), which assumes mutual knowledge of conjectures, yielding consistency.

## 2.3 Nash Equilibrium

Consider the battle of the sexes

| Man\Woman | opera | ballet |
|---|---|---|
| opera | 1,4 | 0,0 |
| ballet | 0,0 | 4,1 |

In this game, there is no dominant strategy. But suppose W is playing opera. Then, the best thing M can do is to play opera, too. Thus opera is a best-response for M against opera. Similarly, opera is a best-response for W against opera. Thus, at (opera, opera), neither party wants to take a different action. This is a Nash Equilibrium.

More formally:

**Definition 12** *For any player $i$, a strategy $s_i^{BR}$ is a* best response *to $s_{-i}$ if and only if*

$$u_i(s_i^{BR}, s_{-i}) \geq u_i(s_i, s_{-i}), \forall s_i \in S_i$$

This definition is identical to that of a dominant strategy except that it is not for all $s_{-i} \in S_{-i}$ but for a specific strategy $s_{-i}$. If it were true for all $s_{-i}$, then $S_i^{BR}$ would also be a dominant strategy, which is a stronger requirement than being a best response against some strategy $s_{-i}$.

**Definition 13** *A strategy profile $(s_1^{NE}, ...s_N^{NE})$ is a* Nash Equilibrium *if and only if $s_i^{NE}$ is a best-response to $s_{-i}^{NE} = (s_1^{NE}, ...s_{i-1}^{NE}, s_{i+1}^{NE}, ...s_N^{NE})$ for each $i$. That is, for all $i$, we have that*

$$u_i(s_i^{NE}, s_{-i}^{NE}) \geq u_i(s_i, s_{-i}^{NE}) \quad \forall s_i \in S_i.$$

In other words, no player would have an incentive to deviate, if he knew which strategies the other players play.

If a strategy profile is a dominant strategy equilibrium, then it is also a NE, but the reverse is not true. For instance, in the Battle of the Sexes, both (O,O) and (B,B) are Nash equilibria, but neither are dominant strategy equilibria. Furthermore, a dominant strategy equilibrium is unique, but as the Battle of the Sexes shows, Nash equilibrium is not unique in general.

**At this point you should stop, and compute the Nash equilibrium in Cournot Duopoly game!!** Why does Nash equilibrium coincide with the rationalizable strategies. In general: Are all rationalizable strategies Nash equilibria? Are all Nash equilibria rationalizable? You should also compute the Nash equilibrium in Cournot oligopoly, Bertrand duopoly and in the commons problem.

The definition above covers only the pure strategies. We can define the Nash equilibrium for mixed strategies by changing the pure strategies with the mixed strategies. Again given the mixed strategy of the others, each agent maximizes his expected payoff over his own (mixed) strategies.[5]

**Example** Consider the Battle of the Sexes again where we located two pure strategy equilibria. In addition to the pure strategy equilibria, there is a mixed strategy equilibrium.

| Man\Woman | opera | ballet |
|---|---|---|
| opera | 1,4 | 0,0 |
| ballet | 0,0 | 4,1 |

Let's write $q$ for the probability that M goes to opera; with probability $1-q$, he goes to ballet. If we write $p$ for the probability that W goes to opera, we can compute her

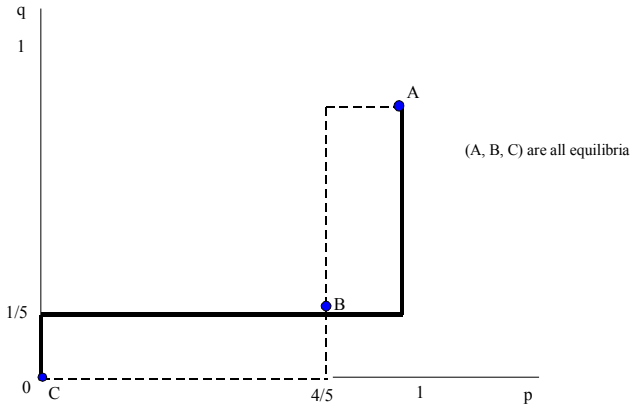
[5] In terms of beliefs, this correspondes to the requirement that, if $i$ assigns positive probability to the event that $j$ may play a particular pure strategy $s_j$, then $s_j$ must be a best response given $j$'s beliefs.

expected utility from this as

$$\begin{aligned} U_2(p;q) &= pqu_2(\text{opera,opera}) + p(1-q)u_2(\text{ballet,opera}) \\ &\quad + (1-p)qu_2(\text{opera,ballet}) + (1-p)(1-q)u_2(\text{ballet,ballet}) \\ &= p[qu_2(\text{opera,opera}) + (1-q)u_2(\text{ballet,opera})] \\ &\quad + (1-p)[qu_2(\text{opera,ballet}) + (1-q)u_2(\text{ballet,ballet})] \\ &= p[q4 + (1-q)0] + (1-p)[0q + 1(1-q)] \\ &= p[4q] + (1-p)[1-q]. \end{aligned}$$

Note that the term $[4q]$ multiplied with $p$ is her expected utility from going to opera, and the term multiplied with $(1-p)$ is her expected utility from going to ballet. $U_2(p;q)$ is strictly increasing with $p$ if $4q > 1-q$ (i.e., $q > 1/5$); it is strictly decreasing with $p$ if $4q < 1-q$, and is constant if $4q = 1-q$. In that case, W's best response is $p = 1$ of $q > 1/5$, $p = 0$ if $q < 1/5$, and $p$ is any number in $[0,1]$ if $q = 1/5$. In other words, W would choose opera if her expected utility from opera is higher, ballet if her expected utility from ballet is higher, and can choose any of opera or ballet if she is indifferent between these two.

Similarly we compute that $q = 1$ is best response if $p > 4/5$; $q = 0$ is best response if $p < 4/5$; and any $q$ can be best response if $p = 4/5$. We plot the best responses in the following graph.




The Nash equilibria are where these best responses intersect. There is one at (0,0), when they both go to ballet, one at (1,1), when they both go to opera, and there is one at (4/5,1/5), when W goes to opera with probability 4/5, and M goes to opera with probability 1/5.

*Note how we compute the mixed strategy equilibrium (for 2x2 games). We choose 1's probabilities so that 2 is indifferent between his strategies, and we choose 2's probabilities so that 1 is indifferent.*

# 14.12 Game Theory—Notes on Theory Rationalizability

Muhamet Yildiz

What are the implications of rationality and players' knowledge of payoffs? What can we infer more if we also assume that players know that the other players are rational? What is the limit of predictive power we obtain as we make more and more assumptions about players' knowledge about the fellow players' rationality? These notes try to explore these questions.

## 1 Rationality and Dominance

We say that a player is rational if and only if he maximizes the expected value of his payoffs (given his beliefs about the other players' play.) For example, consider the following game.

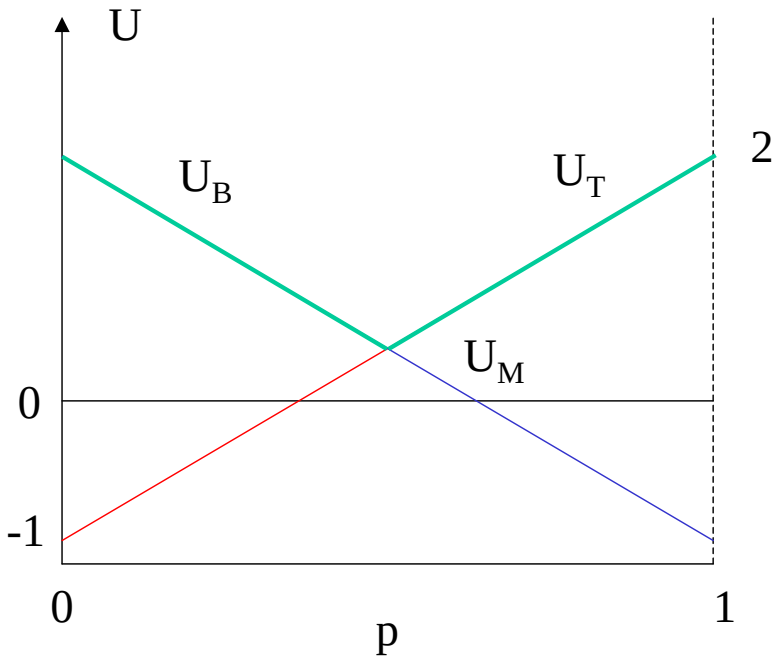
| 1\2 | L | R |
|---|---|---|
| T | 2,0 | -1,1 |
| M | 0,10 | 0,0 |
| B | -1,-6 | 2,0 |

(1)

Consider Player 1. He is contemplating about whether to play T, or M, or B. A quick inspection of his payoffs reveals that his best play depends on what he thinks the other player does. Let's then write $p$ for the probability he assigns to L (as Player 2's play.) Then, his expected payoffs from playing T, M, and B are

$$\begin{aligned} U_T &= 2p - (1-p) = 3p - 1, \\ U_M &= 0, \\ U_B &= -p + 2(1-p) = 2 - 3p, \end{aligned}$$

respectively. These values as a function of $p$ are plotted in the following graph:




As it is clear from the graph, $U_T$ is the largest when $p > 1/2$, and $U_B$ is the largest when $p < 1/2$. At $p = 1/2$, $U_T = U_B > 0$. Hence, if player 1 is rational, then he will play B when $p < 1/2$, D when $p > 1/2$, and B or D if $p = 1/2$. Notice that, if Player 1 is rational, then he will never play M—no matter what he believes about the Player 2's play. Therefore, if we assume that Player 1 is rational (and that the game is as it is described above), then we can conclude that Player 1 will not play M. This is because M is a strictly dominated strategy, a concept that we define now.

Let us use the notation $s_{-i}$ to mean the list of strategies $s_j$ played by all the players $j$ other than $i$, i.e.,

$$s_{-i} = (s_1, ...s_{i-1}, s_{i+1}, ...s_n).$$

**Definition 1** *A strategy* $s_i^*$ strictly dominates $s_i$ *if and only if*

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i}.$$

That is, no matter what the other players play, playing $s_i^*$ is strictly better than playing $s_i$ for player $i$. In that case, if $i$ is rational, he would never play the strictly dominated strategy $s_i$.[1] A mixed strategy $\sigma_i$ dominates a strategy $s_i$ in a similar way:

[1] That is, there is no belief under which he would play $s_i$. Can you prove this?

$\sigma_i$ *strictly dominates* $s_i$ if and only if

$$\sigma_i(s_{i1})u_i(s_{i1}, s_{-i}) + \sigma_i(s_{i2})u_i(s_{i2}, s_{-i}) + \cdots \sigma_i(s_{ik})u_i(s_{ik}, s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i}.$$

Notice that neither of the pure strategies T, M, and B dominates any strategy. Nevertheless, M is dominated by the mixed strategy that $\sigma_1$ that puts probability 1/2 on each of T and B. For each $p$, the payoff from $\sigma_1$ is

$$U_{\sigma_1} = \frac{1}{2}(3p - 1) + \frac{1}{2}(2 - 3p) = \frac{1}{2},$$

which is larger than 0, the payoff from M. As an exercise, one can show that a rational player $i$ will never play a strategy $s_i$ iff $s_i$ is dominated by a (mixed or pure) strategy.

**To sum up:** *if we assume that players are rational (and that the game is as described), then we conclude that no player plays a strategy that is strictly dominated (by some mixed or pure strategy), and this is all we can conclude.*

Although there are few strictly dominated strategies—and thus we can conclude little from the assumption that players are rational—in general, there are interesting games in which the little assumption can lead us to sometimes counterintuitive conclusions. For example, consider the well-known Prisoners' Dilemma game:

| 1\2 | confess | don't confess |
|---|---|---|
| confess | -5,-5 | 0,-6 |
| don't confess | -6,0 | -1,-1 |

Clearly, "don't confess" is strictly dominated by confess, and hence we expect each player to confess, assuming that the game is as described and players are rational.

# 2 Rationalizability or Iterative elimination of strictly dominated strategies

We now want to understand the implications of the assumption that players know that the other players are also rational. To be concrete consider the game in (1). Now, rationality of player 1 requires that he does not play M. For Player 2, her both actions can be a best reply. If she thinks that Player 1 is not likely to play M, then she must play R, and if she thinks that it is very likely that Player 1 will play M, then she must

play L. Hence, rationality of player 2 does not put any restriction on her behavior. But, what if she thinks that it is very likely that player 1 is rational (and that his payoff are as in (1)? In that case, since a rational player 1 does not play M, she must assign very small probability for player 1 playing M. In fact, if she knows that player 1 is rational, then she must be sure that he will not play M. In that case, being rational, she must play R. In summary, *if player 2 is rational and she knows that player 1 is rational, then she must play R.*

Notice that we first eliminated all of the strategies that are strictly dominated (namely M), then taking the resulting game, we eliminated again all of the strategies that are strictly dominated (namely L). This is called *twice iterated elimination of strictly dominated strategies.*

**General fact:** *If a player* (i) *is rational and* (ii) *knows that the other players are also rational (and the payoffs are as given), then he must play a strategy that survives twice iterated elimination of strictly dominated strategies.*

As we impose further assumptions about rationality, we keep iteratively eliminating all strictly dominated strategies (if there remains any). Let's go back to our example in (1). Recall that rationality of player 1 requires him to play T or B, and knowledge of the fact that player 2 is also rational does not put any restriction on his behavior—as rationality itself does not restrict Player 2's behavior. Now, assume that Player 1 also knows (i) that Player 2 is rational and (ii) that Player 2 knows that Player 1 is rational (and that the game is as in (1)). Then, as the above analysis knows, Player 1 must know that Player 2 will play R. In that case, being rational he must play B.

This analysis gives us a mechanical procedure to analyze the games, $n$-times *Iterated Elimination of Strictly Dominated Strategies*: eliminate all the strictly dominated strategies, and iterate this $n$ times.

**General fact:** *If* (1) *every player is rational,* (2) *every player knows that every player is rational,* (3) *every player knows that every player knows that every player is rational, ... and* ($n$) *every player knows that every player knows that ... every player is rational, then every player must play a strategy that survives $n$-times iterated elimination of strictly dominated strategies.*

**Caution**: we do eliminate the strategies that are dominated by some mixed strategies!

Notice that when there are only finitely many strategies, this elimination process must stop at some $n$, i.e., there will be no dominated strategy to eliminate.

**Definition 2** *We call the elimination process that keeps iteratively eliminating all strictly dominated strategies until there is no strictly dominated strategy* Iterated Elimination of Strictly Dominated Strategies*; we eliminate indefinitely if the process does not stop. We call a strategy* rationalizable *if and only if it survives iterated elimination of strictly dominated strategies.*

Clearly,

**General fact:** *If it is common knowledge that every player is rational (and the game is as described), then every player must play a rationalizable strategy. Moreover, any rationalizable strategy is consistent with common knowledge of rationality.*

A problem is there are usually too many rationalizable strategies; the elimination process usually stops too early. In that case a researcher cannot make much prediction based on such analysis. An equally crucial problem is that elimination process may be too slow so that many strategies are eliminated at high iterations. In that case, predictions based on rationalizability will heavily rely on strong assumptions about rationality, i.e., everybody knows that everybody knows that ... everybody is rational.

# 3 Problem description

In the class, we will apply notion of rationalizability for

1. Cournot oligopoly,
2. Bertrand (price) competition, and
3. partnership games.

# 14.12 Game Theory — Supplementary notes on Partnership Games

Muhamet Yildiz

## 1 Intrroduction

Many relationships can be taken as a partnership in which two partners provide an input towards an outcome that they will share. For example, in a firm, the employer and the worker provide capital, know-how, and effort to produce a good that will generate a revenue, which will be divided between the employer and the worker according to the parties' relative bargaining power, or an existing contract that reflects these bargaining powers. Many times, the generated value depends on the parties' inputs in a way that is not additively separable. For example, the marginal increase in the revenue as we change the worker's effort level depends on the level of capital and the machinery employed by the firm. Similarly, the value of an additional machine depends on the effort level of the worker. In most cases, there is a synergy between the parties, i.e., a party's marginal input is more valuable when the other party provides higher input.

This is also true in a grander level when the firms and the workers choose the technologies and the education levels. In an environment with few skilled workers who can operate computers, computerization of the production process will have low return as the firm will have to pay a lot for few skilled workers or employ low-skill workers who will not be able to utilize the computerized system skillfully. Similarly, the value of higher education will be low when there are few firms that can utilize the skills obtained in higher education. A historical example of this is provided by Acemoglu (1998). High school enrollment and graduation rates doubled in the 1910s, mostly due to changes in the location and curricula of schools and the decline in transport costs. The skill premium (white collar wage relative to blue collar wage) fell sharply in the 1910s. Yet, despite the even faster increase in the supply of high school skills during the 1920s, the skill premium levelled off and started a mild increase. Similarly, in 1970s college graduates earned 55 percent more than high school graduates. This premium fell to 41 percent in 1980, but then increased to 62 percent in 1995. As an explanation of this pattern, Acemoglu illustrates that, "when there are more skilled workers, the market for

skill-complementary technologies is larger. Therefore, the inventor will be able to obtain higher profits, and more effort will be devoted to the invention of skill complementary technologies," which will increase the value of skilled workers.

Similar relationship exists when the firms have strategic alliances, or produce complementary goods. For example, the value of a computer depends on both the quality of the operating system (along with other software) and the CPU (and other hardware). More able software requires large memory and fast computation, and such equipment is valuable most when there are software that can utilize such high capacity. Since each firm's profit comes from the sale of computers (which embody both operating system and hardware), they will have a partnership relationship as described above.

Similar partnership games are played by different departments within a firm. For example, both R&D and marketing departments (and their managers) of a firm get shares from the sale of the product, while the sales depends both on the quality of the product (which presumably is increasing with the effort exerted by the researchers in R&D) and the creative advertisement. Once again, the marginal value of better advertisement increases with higher product quality, and vice verse.

I will now formalize such a partnership relationship as a formal game and compute its Nash equilibria.

# 2 Formulation

We have two players, E and W, denoting an employer and a worker. Simultaneously E and W choose $K \in [0, \infty)$ and $L \in [0, \infty)$, which yields an output level $f(K, L)$. To formalize the notion that there is a synergy between the employer and the worker, we assume that $f$ is "supermodular," i.e., given any $K$, $K'$, $L$, $L'$ with $K > K'$ and $L > L'$, we have

$$f(K, L) - f(K', L) > f(K, L') - f(K', L').$$

In particular, we will assume a functional form

$$f(K, L) = K^{\alpha} L^{\beta} \quad (\alpha, \beta, \alpha + \beta \in (0, 1)),$$

which satisfies the above condition. We will assume that the parties share the output equally and the costs of their inputs are linear so that the payoffs of E and W are

$$U_E = \frac{1}{2} K^{\alpha} L^{\beta} - K \quad \text{and} \quad U_W = \frac{1}{2} K^{\alpha} L^{\beta} - L,$$

respectively. We will assume that all these are common knowledge, so that the both strategy spaces are $[0, \infty)$ and the payoff functions are $U_E$ and $U_W$.

# 3 Nash Equilibrium

In order to compute the Nash equilibrium we firs compute the best reply function of each player. Given any $L$, $U_E(K, L)$ is maximized (as a function of $K$) at $K^*$ that is the unique solution to the first order condition that

$$0 = \frac{\partial U_E}{\partial K} = \frac{\alpha}{2} K^{\alpha-1} L^{\beta} - 1,$$

which yields

$$K^* = \left(\frac{\alpha}{2} L^{\beta}\right)^{1/(1-\alpha)}.$$

Similarly, the best reply $L^*$ of W against any given $K$ is

$$L^* = \left(\frac{\beta}{2} K^{\alpha}\right)^{1/(1-\beta)}.$$

These functions are plotted in Figure 1. A Nash equilibrium is any $(K^*, L^*)$ that simultaneously solves the last two equations. Graphically these solutions are the intersections of the graphs of best reply functions. Clearly there are two Nash equilibria one is given by $K = L = 0$, and the other is given by

$$\begin{aligned} K &= \left(\frac{\alpha^{1-\beta}\beta^{\alpha}}{2}\right)^{1/(1-\alpha-\beta)} \\ L &= \left(\frac{\alpha^{\beta}\beta^{1-\alpha}}{2}\right)^{1/(1-\alpha-\beta)}. \end{aligned}$$

# References


[1] Acemoglu, Daron (1998): "Why Do New Technologies Complement Skills? Directed Technical Change and Wage Inequality," *Quarterly Journal of Economics*, volume 113, pp. 1055-1089.

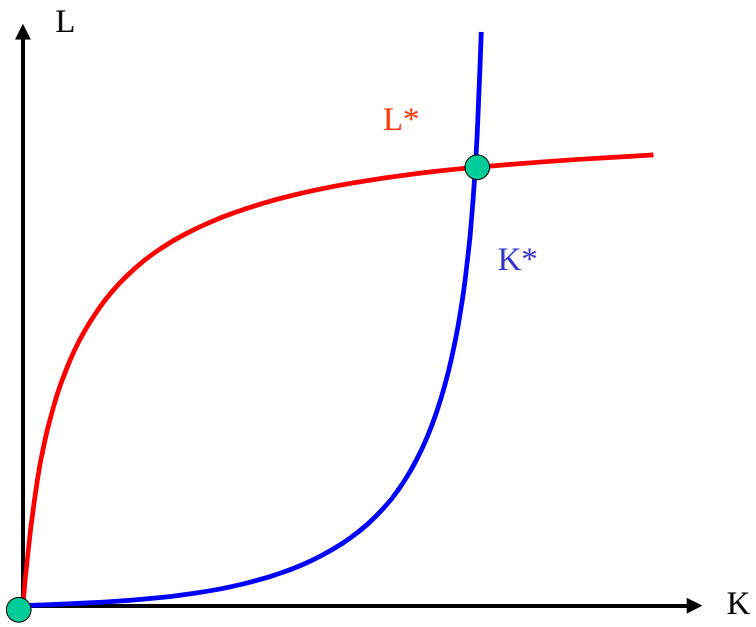



Figure 1:

# 14.12 Game Theory Lecture Notes*
# Lectures 7-9

Muhamet Yildiz

In these lectures we analyze dynamic games (with complete information). We first analyze the perfect information games, where each information set is singleton, and develop the notion of backwards induction. Then, considering more general dynamic games, we will introduce the concept of the subgame perfection. We explain these concepts on economic problems, most of which can be found in Gibbons.

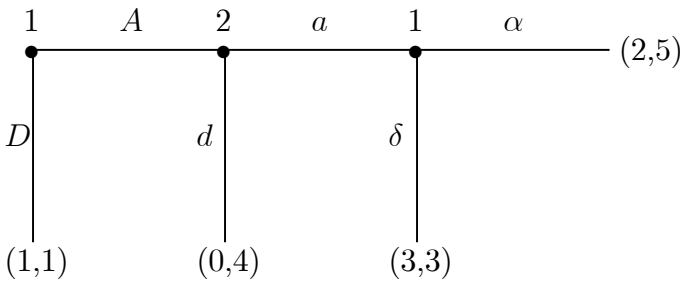
## 1 Backwards induction

The concept of backwards induction corresponds to the assumption that it is common knowledge that each player will act rationally at each node where he moves – even if his rationality would imply that such a node will not be reached.[1] Mechanically, it is computed as follows. Consider a finite horizon perfect information game. Consider any node that comes just before terminal nodes, that is, after each move stemming from this node, the game ends. If the player who moves at this node acts rationally, he will choose the best move for himself. Hence, we select one of the moves that give this player the highest payoff. Assigning the payoff vector associated with this move to the node at hand, we delete all the moves stemming from this node so that we have a shorter game, where our node is a terminal node. Repeat this procedure until we reach the origin.

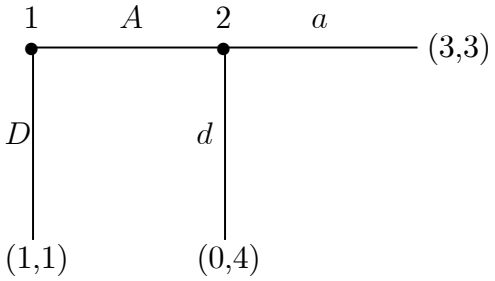
*These notes do not include all the topics that will be covered in the class. See the slides and supplementary notes for a more complete picture.

[1]More precisely: at each node $i$ the player is certain that all the players will act rationally at all nodes $j$ that follow node $i$; and at each node $i$ the player is certain that at each node $j$ that follows node $i$ the player who moves at $j$ will be certain that all the players will act rationally at all nodes $k$ that follow node $j$,...ad infinitum.

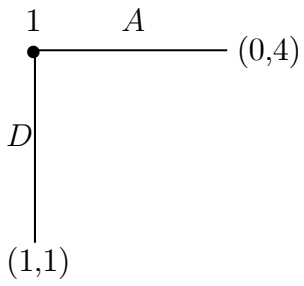
**Example** Consider the following well-known game, called as *the centipedes game*. This game illustrates the situation where it is mutually beneficial for all players to stay in a relationship, while a player would like to exit the relationship, if she knows that the other player will exit in the next day.



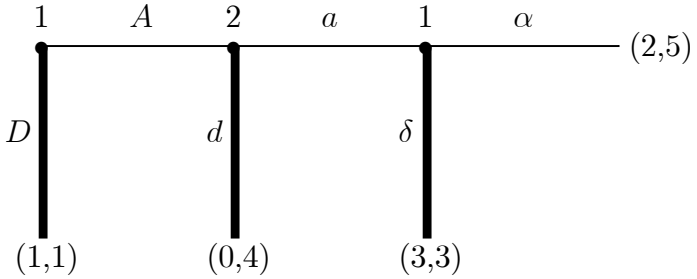

In the third day, player 1 moves, choosing between going across ($\alpha$) or down ($\delta$). If he goes across, he would get 2; if he goes down, he will get 3. Hence, we reckon that he will go down. Therefore, we reduce the game as follows:




In the second day, player 2 moves, choosing between going across ($a$) or down ($d$). If she goes across, she will get 3; if she goes down, she will get 4. Hence, we reckon that she will go down. Therefore, we reduce the game further as follows:




Now, player 1 gets 0 if he goes across ($A$), and gets 1 if he goes down ($D$). Therefore, he goes down. The equilibrium that we have constructed is as follows:



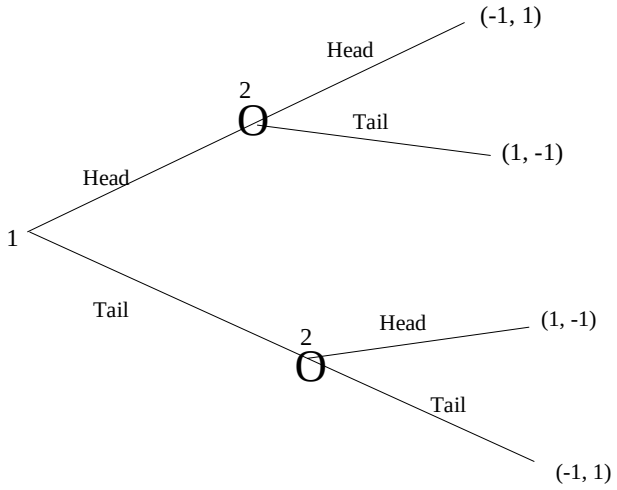

That is, at each node, the player who is to move goes down, exiting the relationship.

Let's go over the assumptions that we have made in constructing our equilibrium. We assumed that player 1 will act rationally at the last date, when we reckoned that he goes down. When we reckoned that player 2 goes down in the second day, we assumed that player 2 assumes that player 1 will act rationally on the third day, and also assumed that she is rational, too. On the first day, player 1 anticipates all these. That is, he is assumed to know that player 2 is rational, and that she will keep believing that player 1 will act rationally on the third day.

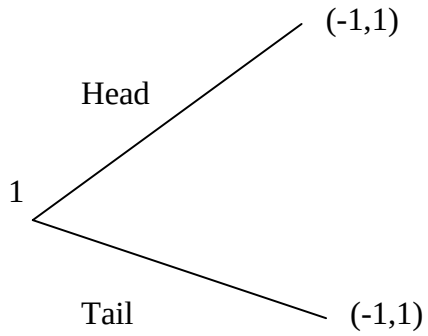
This example also illustrates another notion associated with backwards induction – commitment (or the lack of commitment). Note that the outcomes on the third day (i.e., (3,3) and (2,5)) are both strictly better than the equilibrium outcome (1,0). But they cannot reach these outcomes, because player 2 cannot commit to go across, and anticipating that player 2 will go down, player 1 exits the relationship in the first day. There is also a further commitment problem in this example. If player 1 where able

to commit to go across on the third day, then player 2 would definitely go across on the second day. In that case, player 1 would go across on the first. Of course, player 1 cannot commit to go across on the third day, and the game ends in the first day, yielding the low payoffs (1,0).

As another example, let us apply backwards induction to the Matching Pennies with Perfect Information:




If player 1 chooses Head, player 2 will Head; and if 1 chooses Tail, player 2 will prefer Tail, too. Hence, the game is reduced to




In that case, Player 1 will be indifferent between Head and Tail, choosing any of these two option or any randomization between these two acts will give us an equilibrium with backwards induction.

**At this point, you should stop and study the Stackelberg duopoly in Gibbons.** You should also check that there is also a Nash equilibrium of this game in which

the follower produces the Cournot quantity irrespective of what the leader produces, and the leader produces the Cournot quantity. Of course, this is not consistent with backwards induction: when the follower knows that the leader has produced the Stackelberg quantity, he will change his mind and produce a lower quantity, the quantity that is computed during the backwards induction. For this reason, we say that this Nash equilibrium is based on a non-credible threat (of the follower).

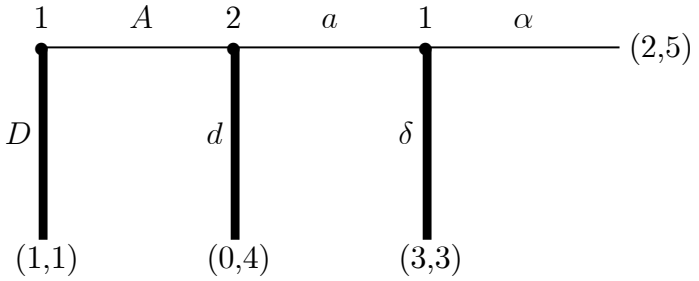
Backwards induction is a powerful solution concept with some intuitive appeal. Unfortunately, we cannot apply it beyond perfect information games with a finite horizon. Its intuition, however, can be extended beyond these games through subgame perfection.

# 2 Subgame perfection

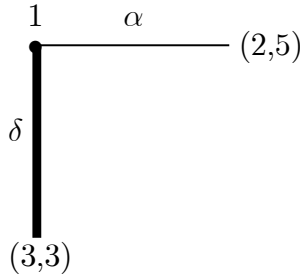
A main property of backwards induction is that, when we confine ourselves to a subgame of the game, the equilibrium computed using backwards induction remains to be an equilibrium (computed again via backwards induction) of the subgame. Subgame perfection generalizes this notion to general dynamic games:

**Definition 1** *A Nash equilibrium is said to be* subgame perfect *if an only if it is a Nash equilibrium in every subgame of the game.*

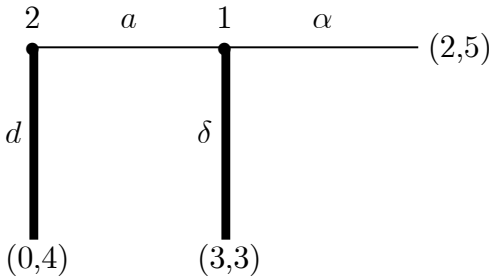
What is a subgame? In any given game, there may be some smaller games embedded; we call each such embedded game a subgame. Consider, for instance, the centipedes game (where the equilibrium is drawn in thick lines):




This game has three subgames. Here is one subgame:



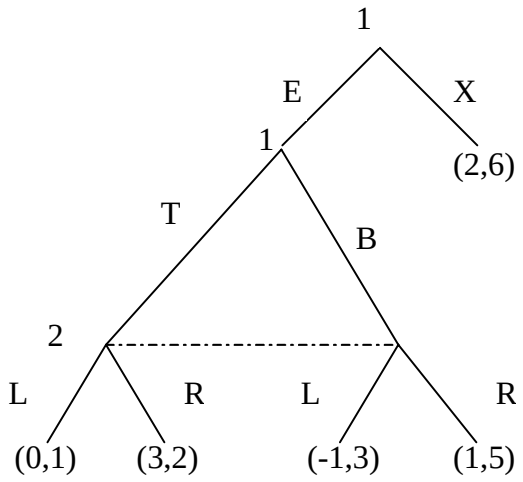

This is another subgame:




And the third subgame is the game itself. We call the first two subgames (excluding the game itself) proper. Note that, in each subgame, the equilibrium computed via backwards induction remains to be an equilibrium of the subgame.

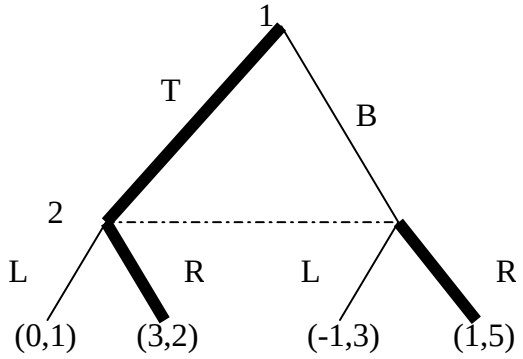
Now consider the matching penny game with perfect information. In this game, we have three subgames: one after player 1 chooses Head, one after player 1 chooses Tail, and the game itself. Again, the equilibrium computed through backwards induction is a Nash equilibrium at each subgame.

Now consider the following game.

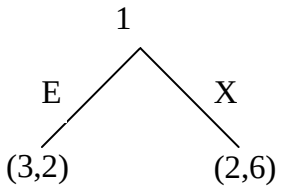



We cannot apply backwards induction in this game, because it is not a perfect information game. But we can compute the subgame perfect equilibrium. This game has two subgames: one starts after player 1 plays E; the second one is the game itself. We compute the subgame perfect equilibria as follows. We first compute a Nash equilibrium of the subgame, then fixing the equilibrium actions as they are (in this subgame), and taking the equilibrium payoffs in this subgame as the payoffs for entering in the subgame, we compute a Nash equilibrium in the remaining game.

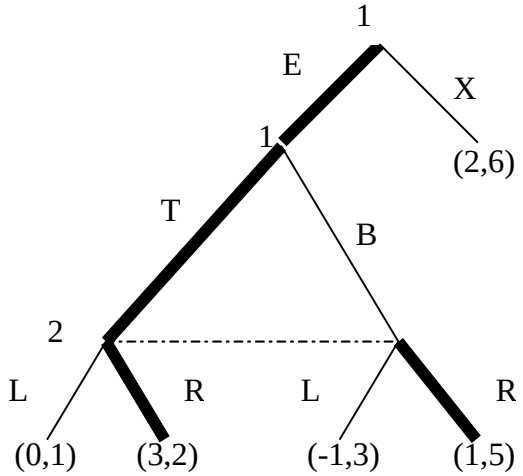
The subgame has only one Nash equilibrium, as T dominates B: Player 1 plays T and 2 plays R, yielding the payoff vector (3,2).



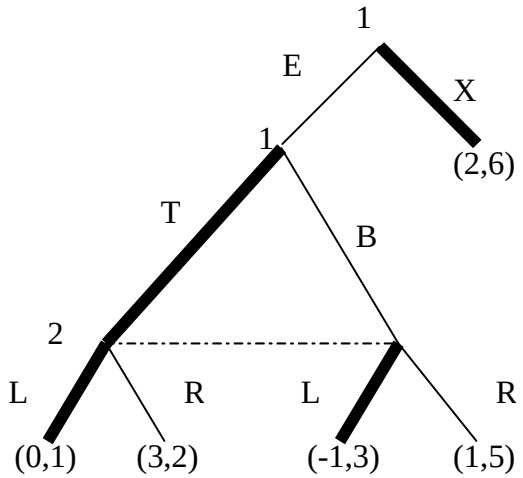

Given this, the remaining game is




where player 1 chooses E. Thus, the subgame-perfect equilibrium is as follows.

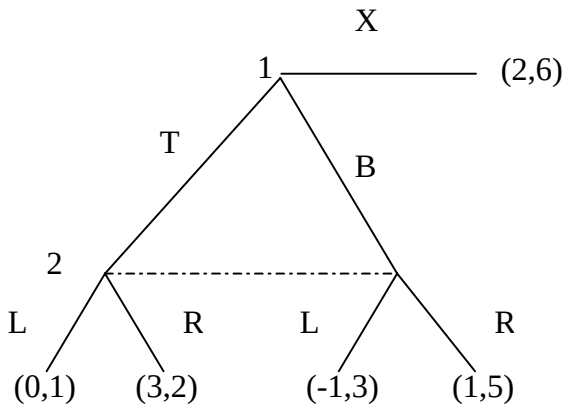



Note that there are other Nash Equilibria; one of them is depicted below.

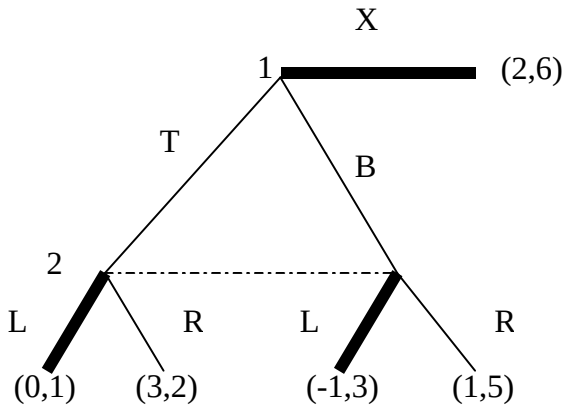



You should be able to check that this is a Nash equilibrium. But it is not subgame perfect, for, in the proper subgame, 2 plays a strictly dominated strategy.

Now, consider the following game, which is essentially the same game as above, with a slight difference that here player 1 makes his choices at once:




Note that the only subgame of this game is itself, hence any Nash equilibrium is subgame perfect. In particular, the non-subgame-perfect Nash equilibrium of the game above is subgame perfect. In the new game it takes the following form:




**At this point you should stop reading and study "tariffs and imperfect international competition".**

# 3 Sequential Bargaining

Imagine that two players own a dollar, which they can use only after they decide how to divide it. Each player is risk-neutral and discounts the future exponentially. That is, if a player gets $x$ dollar at day $t$, his payoff is $\delta^t x$ for some $\delta \in (0,1)$. The set of all feasible divisions is $D = \left\{(x,y) \in [0,1]^2 \,|x+y \leq 1\right\}$. Consider the following scenario. In the first day player one makes an offer $(x_1, y_1) \in D$. Then, knowing what has been offered,

player 2 accepts or rejects the offer. If he accepts the offer, the offer is implemented, yielding payoffs $(x_1, y_1)$. If he rejects the offer, then they wait until the next day, when player 2 makes an offer $(x_2, y_2) \in D$. Now, knowing what player 2 has offered, player 1 accepts or rejects the offer. If player 1 accepts the offer, the offer is implemented, yielding payoffs $(\delta x_2, \delta y_2)$. If player two rejects the offer, then the game ends, when they lose the dollar and get payoffs (0,0).

Let us analyze this game. On the second day, if player 1 rejects the offer, he gets 0. Hence, he accepts any offer that gives him more than 0, and he is indifferent between accepting and rejecting any offer that gives him 0. Assume that he accepts the offer (0,1).[2] Then, player 2 would offer (0,1), which is the best player 2 can get. Therefore, if they do not agree on the first day, then player 2 takes the entire dollar on the second day, leaving player 1 nothing. The value of taking the dollar on the next day for player 2 is $\delta$. Hence, on the first day, player 2 will accept any offer that gives him more than $\delta$, will reject any offer that gives him less than $\delta$, and he is indifferent between accepting and rejecting any offer that gives him $\delta$. As above, assume that player 2 accepts the offer $(1-\delta, \delta)$. In that case, player 1 will offer $(1-\delta, \delta)$, which will be accepted. For any division that gives player 1 more than $1-\delta$ will give player 2 less than $\delta$, and will be rejected.

Now, consider the game in which the game above is repeated $n$ times. That is, if they have not yet reached an agreement by the end of the second day, on the third day, player 1 makes an offer $(x_3, y_3) \in D$. Then, knowing what has been offered, player 2 accepts or rejects the offer. If he accepts the offer, the offer is implemented, yielding payoffs $\left(\delta^2 x_3, \delta^2 y_3\right)$. If he rejects the offer, then they wait until the next day, when player 2 makes an offer $(x_4, y_4) \in D$. Now, knowing what player 2 has offered, player 1 accepts or rejects the offer. If player 1 accepts the offer, the offer is implemented, yielding payoffs $\left(\delta^3 x_4, \delta^3 y_4\right)$. If player two rejects the offer, then they go to the 5th day... And this goes on like this until the end of day $2n$. If they have not yet agreed at the end of that day, the game ends, when they lose the dollar and get payoffs (0,0).

The subgame perfect equilibrium will be as follows. At any day $t = 2n - 2k$ ($k$ is a

[2] In fact, player 1 must accept (0,1) in equilibrium. For, if he doesn't accept (0,1), the best response of player 2 will be empty, inconsistent with an equilibrium. (Any offer $(\epsilon, 1-\epsilon)$ of player 2 will be accepted. But for any offer $(\epsilon, 1-\epsilon)$, there is a better offer $(\epsilon/2, 1-\epsilon/2)$, which will also be accepted.)

non-negative integer), player 1 accepts any offer $(x, y)$ with

$$x \geq \frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}$$

and will reject any offer $(x, y)$ with

$$x < \frac{\delta\left(1-\delta^{2k}\right)}{1+\delta};$$

and player 2 offers

$$(x_t, y_t) = \left(\frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}, 1 - \frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}\right) \equiv \left(\frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}, \frac{1+\delta^{2k+1}}{1+\delta}\right).$$

And at any day $t-1 = 2n-2k-1$, player 2 accepts an offer $(x, y)$ iff

$$y \geq \frac{\delta\left(1+\delta^{2k+1}\right)}{1+\delta};$$

and Player 1 will offer

$$(x_{t-1}, y_{t-1}) = \left(1 - \frac{\delta\left(1+\delta^{2k+1}\right)}{1+\delta}, \frac{\delta\left(1+\delta^{2k+1}\right)}{1+\delta}\right) \equiv \left(\frac{1-\delta^{2k+2}}{1+\delta}, \frac{\delta\left(1+\delta^{2k+1}\right)}{1+\delta}\right).$$

We can prove this is the equilibrium given by backwards induction using mathematical induction on $k$. (That is, we first prove that it is true for $k = 0$; then assuming that it is true for some $k-1$, we prove that it is true for $k$.)

**Proof.** Note that for $k = 0$, we have the last two periods, identical to the 2-period example we analyzed above. Letting $k = 0$, we can easily check that the behavior described here is the same as the equilibrium behavior in the 2-period game. Now, assume that, for some $k-1$ the equilibrium is as described above. That is, at the beginning of date $t+1 := 2n - 2(k-1) - 1 = 2n - 2k + 1$, player 1 offers

$$(x_{t+1}, y_{t+1}) = \left(\frac{1-\delta^{2(k-1)+2}}{1+\delta}, \frac{\delta\left(1+\delta^{2(k-1)+1}\right)}{1+\delta}\right) = \left(\frac{1-\delta^{2k}}{1+\delta}, \frac{\delta\left(1+\delta^{2k-1}\right)}{1+\delta}\right);$$

and his offer is accepted. At date $t = 2n-2k$, player one accepts an offer iff the offer is at least as good as having $\frac{1-\delta^{2k}}{1+\delta}$ in the next day, which is worth $\frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}$. Therefore, he will accept an offer $(x, y)$ iff

$$x \geq \frac{\delta\left(1-\delta^{2k}\right)}{1+\delta};$$

as we have described above. In that case, the best player 2 can do is to offer

$$(x_t, y_t) = \left( \frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}, 1 - \frac{\delta\left(1-\delta^{2k}\right)}{1+\delta} \right) = \left( \frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}, \frac{1+\delta^{2k+1}}{1+\delta} \right).$$

For any offer that gives 2 more than $y_t$ will be rejected in which case player 2 will get

$$\delta y_{t+1} = \frac{\delta^2\left(1+\delta^{2k-1}\right)}{1+\delta} < y_t.$$

That is, at $t$ player 2 offers $(x_t, y_t)$; and it is accepted. In that case, at $t-1$, player 2 will accept an offer $(x, y)$ iff

$$y \geq \delta y_t = \frac{\delta\left(1+\delta^{2k+1}\right)}{1+\delta}.$$

In that case, at $t-1$, player 1 will offer

$$(x_{t-1}, y_{t-1}) \equiv (1-\delta y_t, \delta y_t) = \left( \frac{1-\delta^{2k+2}}{1+\delta}, \frac{\delta\left(1+\delta^{2k+1}\right)}{1+\delta} \right),$$

completing the proof. ■

Now, let $n \to \infty$. At any odd date $t$, player 1 will offer

$$(x_t^\infty, y_t^\infty) = \lim_{k\to\infty} \left( \frac{1-\delta^{2k+2}}{1+\delta}, \frac{\delta\left(1+\delta^{2k+1}\right)}{1+\delta} \right) = \left( \frac{1}{1+\delta}, \frac{\delta}{1+\delta} \right);$$

and any even date $t$ player 2 will offer

$$(x_t^\infty, y_t^\infty) = \lim_{k\to\infty} \left( \frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}, \frac{1+\delta^{2k+1}}{1+\delta} \right) = \left( \frac{\delta}{1+\delta}, \frac{1}{1+\delta} \right);$$

and the offers are barely accepted.

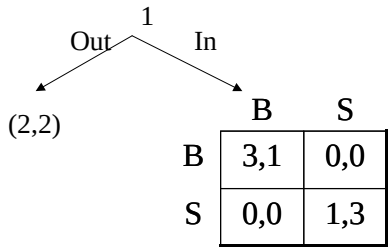
# 14.12 Game Theory Lecture Notes
# Forward Induction

Muhamet Yildiz

Forward induction is a term that is used for the vague idea that when one sees a move by a player, he tries to understand what the player intends to do in the future—even if this move is not supposed to be taken (according to a theory one might have in mind.) Many economists have attempted to formalize this notion. These notes intend to present this idea informally (following a recent formalization by Battigalli and Siniscalchi.)

## 1 Example

In order to see the basic idea, consider the following game.




In this game, player 1 has option of staying out and getting the payoff of 2, rather than playing the battle of sexes game with player 2. Now the battle of sexes has three Nash equilibria: the pure strategy equilibria (B,B) and (S,S), and the mixed strategy equilibrium ((3/4,1/4),(1/4,3/4)), where player 1 (resp., 2) plays strategy B (resp, S) with probability 3/4. These equilibria lead to three subgame-perfect equilibria in the larger game:

1. Player 1 plays In and then they play (B,B);

2. Player 1 plays out, but they would play (S,S) if player 1 played In, and

3. Player 1 plays out, but they would play the mixed-strategy equilibrium ((3/4,1/4),(1/4,3/4)) if player 1 played In.

Let us look at the last two equilibria closely. In these equilibria, after seeing that player 1 has played In, player 2 believes that player 1 will play S with positive probability (namely with probabilities 1 and 1/4 in equilibria in 2 and 3, respectively, above). In other words, after seeing In, player 2 thinks that player 1 plays the strategy InS with positive probability. But notice that this strategy is strictly dominated by staying out (i.e., by the strategies OutB and OutS). Hence, after observing that player 1 has played In, Player 2 comes to think that Player 1 may be irrational. Notice, however, that playing In does not provide any strong evidence for irrationality of Player 1. Player 1 might have played In with the intention of playing B afterwards, thinking that player 2 will also play B. That is, after seeing In, Player 2 has revised his beliefs about the rationality of Player 1, while he could have revised his beliefs about Player 1's intentions and beliefs. That means that he did not believe in the rationality of Player 1 strongly enough. Had he believed in the rationality of player 1 strongly, after seeing In, he would become certain that player 1 will play B, and thus he would also play B. Therefore, if player 1 had sufficient confidence in that player 2 strongly believes that player 1 is rational, then she would anticipate that he would play B, and she would play In. Therefore, the last two equilibria cannot be consistent with the assumptions (i) that players "strongly believe" in players' rationality and (ii) that they are certain that players "strongly believe" in players' rationality.[1]

The argument in the previous paragraph is a froward induction argument, as it is based an the idea that after seeing a move players must try to think about what the other players are trying to do, and interpret these moves as parts of a rational strategy if possible. In this way, forward induction introduces two important notions into the analysis:

1. *Context*: In analyzing a game, one should not take the game in isolation, but should rather determine the larger context in which the game is being played. For example, the analysis of the battle of sexes may change dramatically, once we

[1] But, since they are equilibria, they must be consistent with the common knowledge of rationality. Think about this issue until you see why this may look paradoxical and why we do not have any contradiction.

realize that one of the players had forgone some high payoff in order to play this game. There is a clear tension between this idea and the idea that "bygones are bygones," the idea that is partly embedded in the solution concepts of backwards induction and subgame-perfection.

2. *Communication*: In a dynamic game, a player's moves may also reveal information about his intentions in the future, and a player may try to communicate his intensions through his actions. For example, by playing In, player 1 communicates his intention that he will play B.

# 2 Concepts

**Communication** Consider a dynamic game. At the beginning each player has some beliefs about the players' payoffs and the physical world; he also has a belief about what kind of beliefs the other players may have about this underlying world, and beliefs about other players' beliefs about other players' beliefs and so on. Call this external uncertainty. In addition to this, each player also have some beliefs about what the other players intend to play (as a function their beliefs and payoffs), beliefs about other players' beliefs about how the game will be played, and so on. Call the latter uncertainty epistemic uncertainty. During the play of the game, players observe the actions taken by the other players. Each action of a player reveals some information about

(i) the player's beliefs (and information) regarding the external and epistemic uncertainty, and

(ii) the strategy the player intends to play.

Hence, as the game evolves the players interpret every move and revise their beliefs about the external uncertainty and their theories about how the players will play in the future. Now, knowing all these, sophisticated players must also think (and have a belief) about how each player will interpret any given move. (And, again, they have beliefs about these beliefs, and so on.) Hence, when they take an action, they also take into account what they communicate with their actions. The communication about the

epistemic uncertainty is the subject matter of forward induction. In order to focus on this issue, we will assume that there is no external uncertainty.[2]

Keep in mind that a player has a new set of beliefs at each node—about everything one can think of, including the players' beliefs at all decision nodes. We usually require that a player's beliefs at any two nodes are consistent in the sense that when we reach a new node he updates his beliefs (from his beliefs in the previous node) using the Bayes' rule whenever that rule is applicable.[3]

**Sequential Rationality** Recall that by rationality we usually mean that the player maximizes the expected value of his payoffs (given his beliefs about the other players' play.) This definition does not put any restriction about how a player acts at decision nodes that will not be reached according to the player's beliefs. For forward induction, we will use a stronger notion of rationality, known as sequential rationality, which is also used in backwards induction.

**Definition 1** *We say that a player is* sequentially rational *if and only if, at each information set he is to move, he maximizes his expected utility with respect to his beliefs conditional on that he is at that information set.*

Notice that we require that a player's action at a node (or information set) is optimal even if this node is precluded by his own strategy, or he thinks at the beginning of the game that this node will not be reached. For example, in the example above, if player 1 believes that player 2 will play S, then the only sequentially rational move for her is OutS; at her decision node after In, she must give a best reply to S, which is necessarily S, and at the initial node, she will choose Out and get 2, rather than playing In and getting 1. Notice that, given her beliefs, OutB is rational but not sequentially rational. Similarly, if player 1 believes that player 2 will play B, then the only rational (and hence the only sequentially rational) strategy for her is InB.

[2] This issue has been analyzed in game theory extensively using signaling games, and it is very important in analysis of dynamic games, from bargaining to auctions. Of course, these two issues are closely related.

[3] We also require consistency in another sense, but describing this is beyond the scope of these notes.

**Correct Strong Belief** As it must be clear by now, a player may initially believe in an event but may lose his belief after observing some moves. The moves may contradict his initial beliefs after all. We are interested strong beliefs.

**Definition 2** *We say that a player* strongly believes *in some event (or some proposition), if she keeps believing in the event (or the proposition) at each information set where it is possible that the event (or the proposition) is true.*

As an example, consider the event

$$\text{Player 1 is sequentially rational} \tag{SR1}$$

in the game above. If player 1 strongly believes in this, he must believe at the beginning that player 1 will not play the strictly dominated strategy InS. All the other strategies are consistent with sequential rationality of player 1 at the beginning. Consider his information set after he observes In. There are two possible strategies of player 1 that is consistent with this history: InS and InB. Moreover, InB is consistent with SR1, and InS is not. Hence, if player 2 strongly believes in SR1, then he must believe at this information set that player 1 plays InB.

That someone strongly believes in something does not mean that it is true. For example, it is conceivable that there are two individuals, one strongly believes in the existence of God, while the other strongly believes in the non-existence of God. We face some subtle difficulties when we analyze strong beliefs that happens to be false. Hence, we will consider only the cases that someone believes is some event and that event happens to be true.

**Definition 3** *We say that a player $i$* correctly strongly believes *in some event $E$, denoted by $CSB_i(E)$, iff $E$ is true and he strongly believes that $E$ is true.*

For example, that player 2 correctly, strongly believes in SR1 is equivalent to (i) that player 2 believes in SR1 whenever it is possible and (ii) that it is actually true that Player 1 is sequentially rational. Recall also that this implies that player 1 is sequentially rational, and at his information set, player 2 believes that player 1 plays InB.

**Iterated Correct Strong Belief** Recall the definition of common knowledge: we say that an event $E$ is common knowledge iff $E$ is true and everyone knows that it is true and everyone knows that everyone knows that $E$ is true, ad infinitum. We want to define such an iterated notion for correct strong beliefs. We write $CSB(E)$ to mean that everyone correctly strongly believes that $E$ is true. We can also write $CSB^2(E) = CSB(CSB(E))$ to mean that everybody correctly and strongly believes that everybody correctly and strongly believes that $E$ is true. We can write $CSB^k(E) = CSB\left(CSB^{k-1}(E)\right)$, obtaining an iteration. We can now talk about *common correct strong belief*; when applied to some event $E$, this means that $E$ is true and everyone correctly and strongly believes that $E$ is true and everybody correctly and strongly believes that everybody correctly and strongly believes that $E$ is true, and so on.

To see how these assumptions work, consider $CSB^2(E)$. Let us write $SB(E)$ to mean that everybody strongly believes that $E$ is true, and also write $\cap$ to mean "and". We have

$$\begin{aligned} CSB^2(E) &= CSB(CSB(E)) \\ &= CSB(E) \cap SB(CSB(E)) \\ &= E \cap SB(E) \cap SB(E \cap SB(E)), \end{aligned}$$

where the second and third equalities are obtained by inserting the definition of correct strong belief. Notice that when $CSB^2(E)$ is true,

1. $E$ is true,

2. everyone strongly believes that $E$ is true, that is, at every history where $E$ is not contradicted everyone keeps believing in truthfulness of $E$, and

3. everyone strongly believes that the preceding two propositions are true, i.e., they keep believing in them until it is not possible to believe in both propositions simultaneously.

What happens if we reach a history that does not contradict $E$ but could not be reached if $E$ were true and everyone strongly believed that $E$ is true? Let us look at a player's beliefs at such a history. Since $E$ is not contradicted, he believes that $E$ is true, but he no longer believes that everybody strongly believes that $E$ is true. That is, when

a stronger assumption (i.e., $E \cap SB(E)$) is contradicted, he keeps believing in $E$, until it is not possible. Or consider a history at which it is not possible that $E$ is true and everyone strongly believes in this (i.e., $SB(E)$) and all the other players strongly believe in these two (i.e., $\cap_{j \neq i} SB_j(E \cap SB(E))$), but it is possible that $E \cap SB(E)$. In that history, our player will believe $E \cap SB(E)$ but will not believe in $\cap_{j \neq i} SB_j(E \cap SB(E))$. In summary, he believes in everyone strongly believes that everyone strongly believes that ... everyone strongly believes that $E$ is true as many times as he can.

# 3 Forward Induction

We use common correct strong belief in sequential rationality to formalize forward induction. Consider the assumption

$$\text{Everybody is sequentially rational.} \tag{SR}$$

Consider

$$CCSB(SR) \equiv SR \cap CSB(SR) \cap CSB(CSB(SR)) \cap \cdots,$$

i.e., everybody is sequentially rational, everybody correctly and strongly believes that everybody is sequentially rational, and everybody correctly and strongly believes that everybody correctly and strongly believes that everybody is sequentially rational, ... By forward induction, we mean the implications of this set of assumptions.

In our example, notice that $CSB_2(SR1)$ implies that at his information set, player 2 believes that player 1 will play InB. Hence, $SR2 \cap CSB_2(SR1)$ implies that, if player 1 plays In, player 2 will play S. Now,

$$CSB_1(CSB(SR)) = SR1 \cap SR2 \cap SB_1(SR2) \cap SB_2(SR1) \cap SB_1(SR2 \cap CSB_2(SR1)).$$

Hence, $CSB_1(CSB(SR))$ implies that at the beginning player 1 believes that if she plays In, then player 2 will play S (this comes from $SB_1(SR2 \cap CSB_2(SR1))$). Since $CSB_1(CSB(SR))$ implies $SR1$ (i.e., player 1 is rational), this further implies that player 1 must play In.

# 4 Problem Descriptions

Think about how you can apply this, in games where

- there are entry costs (which may be of the form of opportunity cost, such as the possibility of working for a competitive wage in our partnership game), or
- some of the players have opportunity to wastefully spend some of his wealth (e.g., a candidate spending his champaign contributions in ways that do not necessarily increase his popularity, or a well-known company, such as CocaCola or Pepsi, having excessive advertisements that do not contain much information about its products—and would not presumably increase the name recognition of such a well-known company), or
- the right to participate in the game is auctioned (e.g., the telecommunication companies bid in an auction for FCC licenses, which they need to have in order to be in the telecommunication market, where they also compete with each other).

Think also about the internet markets (such as e-bay and yahoo) where the participants pay a fee. Would the demand for these market be decreasing in the entree fee?

# 14.12 Game Theory Lecture Notes
# Lectures 11-12

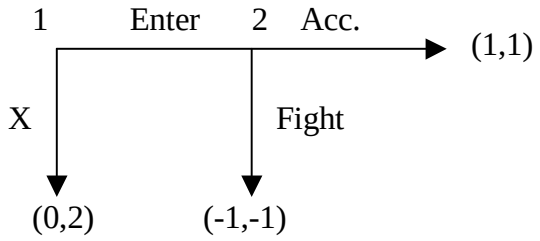
Muhamet Yildiz

## 1 Repeated Games

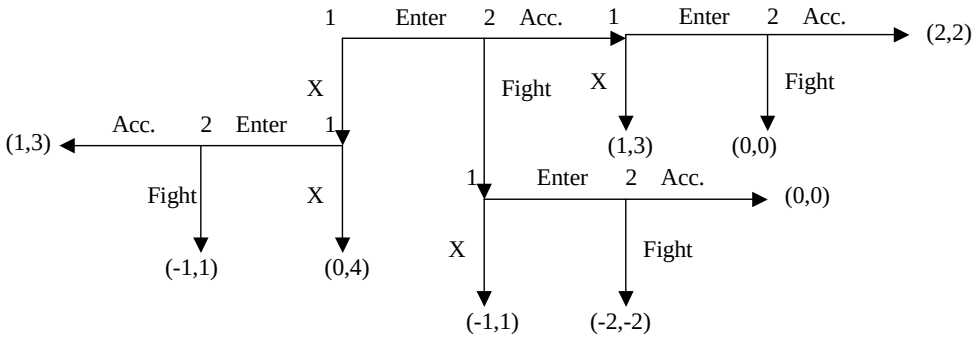
In these notes, we'll discuss the repeated games, the games where a particular smaller game is repeated; the small game is called the stage game. The stage game is repeated regardless of what has been played in the previous games. For our analysis, it is important whether the game is repeated finitely or infinitely many times, and whether the players observe what each player has played in each previous game.

### 1.1 Finitely repeated games with observable actions

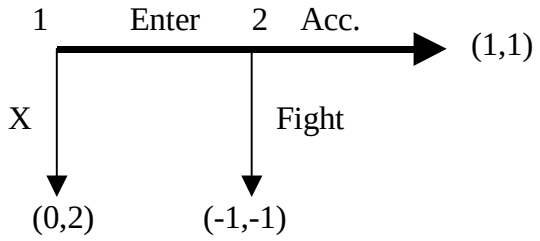
We will first consider the games where a stage game is repeated finitely many times, and at the beginning of each repetition each player recalls what each player has played in each previous play. Consider the following entry deterrence game, where an entrant (1) decides whether to enter a market or not, and the incumbent (2) decides whether to fight or accommodate the entrant if he enters.



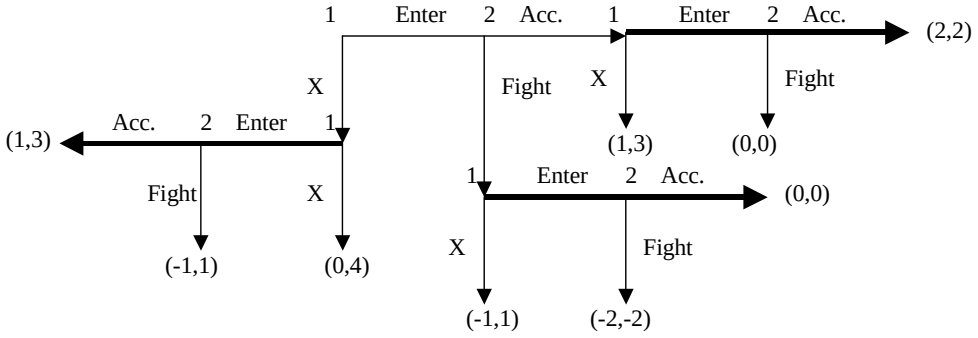

Consider the game where this entry deterrence game is repeated twice, and all the previous actions are observed. Assume that a player simply cares about the sum of his payoffs at the stage games. This game is depicted in the following figure.



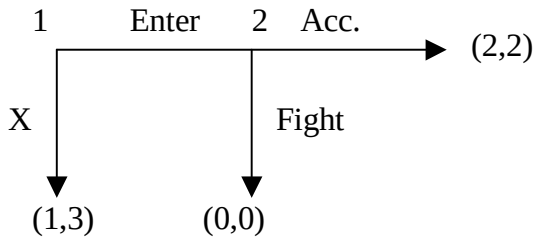

Note that after the each outcome of the first play, the entry deterrence game is played again –where the payoff from the first play is added to each outcome. Since a player's preferences over the lotteries do not change when we add a number to his utility function, each of the three games played on the second "day" is the same as the stage game (namely, the entry deterrence game above). The stage game has a unique subgame perfect equilibrium, where the incumbent accommodates the entrant, and anticipating this, the entrant enters the market.



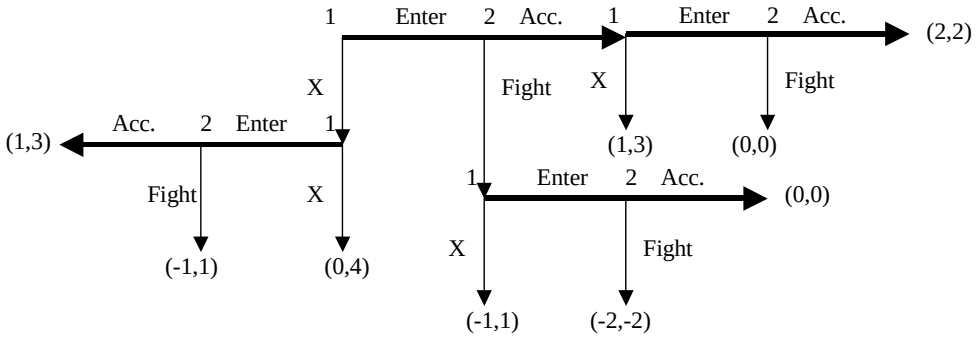

In that case, each of the three games played on the second day has only this equilibrium as its subgame perfect equilibrium. This is depicted in the following.




Using backward induction, we therefore reduce the game to the following.




Notice that we simply added the unique subgame perfect equilibrium payoff of 1 from the second day to each payoff in the stage game. Again, adding a constant to a player's payoffs does not change the game, and hence the reduced game possesses the subgame perfect equilibrium of the stage game as its unique subgame perfect equilibrium. Therefore, the unique subgame perfect equilibrium is as depicted below.




This can be generalized. That is, given any finitely repeated game with observable actions, if the stage game has a unique subgame perfect equilibrium, then the repeated game has a unique subgame perfect equilibrium, where the subgame perfect equilibrium of the stage game is player at each day.

If the stage game has more than one equilibrium, then in the repeated game we may have some subgame perfect equilibria where, in some stages, players play some actions that are not played in any subgame perfect equilibria of the stage game. For the equilibrium to be played on the second day can be conditioned to the play on the first day, in which case the " reduced game" for the first day is no longer the same as the stage game, and thus may obtain some different equilibria. **To see this, see Gibbons.**

## 1.2 Infinitely repeated games with observed actions

Now we consider the infinitely repeated games where all the previous moves are common knowledge at the beginning of each stage. In an infinitely repeated game, we cannot simply add the payoffs of each stage, as the sum becomes infinite. For these games, we will confine

ourselves to the case where players maximize the discounted sum of the payoffs from the stage games. The *present value* of any given payoff stream $\pi = (\pi_0, \pi_1, \ldots, \pi_t, \ldots)$ is computed by

$$PV(\pi; \delta) = \sum_{t=0}^{\infty} \delta^t \pi_t = \pi_0 + \delta\pi_1 + \cdots + \delta^t \pi_t + \cdots,$$

where $\delta \in (0, 1)$ is the *discount factor*. By the *average value*, we simply mean

$$(1 - \delta) PV(\pi; \delta) \equiv (1 - \delta) \sum_{t=0}^{\infty} \delta^t \pi_t.$$

Note that, when we have a constant payoff stream (i.e., $\pi_0 = \pi_1 = \cdots = \pi_t = \cdots$), the average value is simply the stage payoff (namely, $\pi_0$). Note that the present and the average values can be computed with respect to the current date. That is, given any $t$, the present value at $t$ is

$$PV_t(\pi; \delta) = \sum_{s=t}^{\infty} \delta^{s-t} \pi_s = \pi_t + \delta\pi_{t+1} + \cdots + \delta^k \pi_{t+k} + \cdots.$$

Clearly,

$$PV(\pi; \delta) = \pi_0 + \delta\pi_1 + \cdots + \delta^{t-1} \pi_{t-1} + \delta^t PV_t(\pi; \delta).$$

Hence, the analysis does not change whether one uses $PV$ or $PV_t$, but using $PV_t$ is simpler.

The main property of infinitely repeated games is that the set of equilibria becomes very large as players get more patients, i.e., $\delta \to 1$. given any payoff vector that gives each player more than some Nash equilibrium outcome of the stage game, for sufficiently large values of $\delta$, there exists some subgame perfect equilibrium that yields the payoff vector at hand as the average value of the payoff stream. This fact is called the folk theorem. **See Gibbons for details.**

In these games, to check whether a strategy profile $s = (s_1, s_2, \ldots, s_n)$ is a subgame perfect equilibrium, we use the *single-deviation principle*, defined as follows.[1] Take any formation set, where some player $i$ is to move, and play a strategy $a^*$ of the stage game according to the strategy profile $s$. Assume that the information set is reached, each player $j \neq i$ sticks to his strategy $s_j$ in the remaining game, and player $i$ will stick to his strategy $s_i$ in the remaining game except for the information set at hand. given all these, we check whether the player has an incentive to deviate to some action $a'$ at the information set (rather than playing $a^*$). [Note that all players, including player $i$, are assumed to stick to this strategy profile in the remaining game.] The single-deviation principle states that if there is no information set the

[1] Note that a strategy profile $s_i$ is an infinite sequence $s_i = (a_0, a_1, \ldots, a_t, \ldots)$ of functions $a_t$ determining which "strategy of the stage game" to be played at $t$ depending on which actions each player has taken in the previous plays of the stage game.

player has an incentive to deviate in this sense, then the strategy profile is a subgame perfect equilibrium.

Let us analyze the infinitely repeated version of the entry deterrence game. Consider the following strategy profile. At any given stage, the entrant enters the market if an only if the incumbent has accommodated the entrant sometimes in the past. The incumbent accommodates the entrant if an only if he has accommodated the entrant before. (This is a switching strategy, where initially incumbent fights whenever there is an entry and the entrant never enters. If the incumbent happens to accommodate an entrant, they switch to the new regime where the entrant enters the market no matter what the incumbent does after the switching, and incumbent always accommodates the entrant.) For large values of $\delta$, this an equilibrium.

To check whether this is an equilibrium, we use the single-deviation principle. We first take a date $t$ and any history (at $t$) where incumbent has accommodated the entrants. According to the strategy of the incumbent, he will always accommodate the entrant in the remaining game, and the entrant will always enter the market (again according to his own strategy). Thus, the continuation value of incumbent (i.e., the present value of the equilibrium payoff-stream of the incumbent) at $t+1$ is

$$V_A = 1 + \delta + \delta^2 + \cdots = 1/(1-\delta).$$

If he accomodates at $t$, his present value (at $t$) will be $1+\delta V_A$. If he fights, then his present value will be $-1+\delta V_A$. Therefore, the incumbent has no incentive to fight, rather than accomodating as stipulated by his strategy. The entrants continuation value at $t+1$ will also be independent of what happens at $t$, hence the entrant will enter (whence he gets $1 + \delta$[His present value at $t+1$]) rather than deviating (whence he gets $0 + \delta$[His present value at $t+1$]).

Now consider a history at some date $t$ where the incumbent has never accommodated the entrant before. Consider the incumbent's information set. If he accommodates the entrant, his continuation value at $t+1$ will be $V_A = 1/(1-\delta)$, whence his continuation value at $t$ will be $1 + \delta V_A = 1 + \delta/(1-\delta)$. If he fights, however, according to the strategy profile, he will never accommodate any entrants in the future, and the entrant will never enter, in which case the incumbent will get the constant payoff stream of 2, whose present value at$t+1$ is $2/(1-\delta)$. Hence, in this case, his continuation value at $t$ will be $-1+\delta \cdot 2/(1-\delta)$. Therefore, the incumbent will not have any incentive to deviate (and accommodate the entrant) if and only if

$$-1 + \delta \cdot 2/(1-\delta) \geq 1 + \delta/(1-\delta),$$

which is true if and only if

$$\delta \geq 2/3.$$

When this condition holds, the incumbent do not have an incentive to deviate in such histories. Now, if the entrant enters the market, incumbent will fight, and the entrant will never enter in the future, in which case his continuation value will be $-1$. If he does not enter, his continuation value is 0. Therefore, he will not have any incentive to enter, either. Since we have covered all possible histories, by the single-deviation principle, this strategy profile is a subgame perfect equilibrium if and only if $\delta \geq 2/3$.

**Now, study the cooperation in the prisoners' dilemma, implicit collusion in a Cournot duopoly, and other examples in Gibbons.**

# 14.12 Game Theory Lecture Notes
# Lectures 15-18

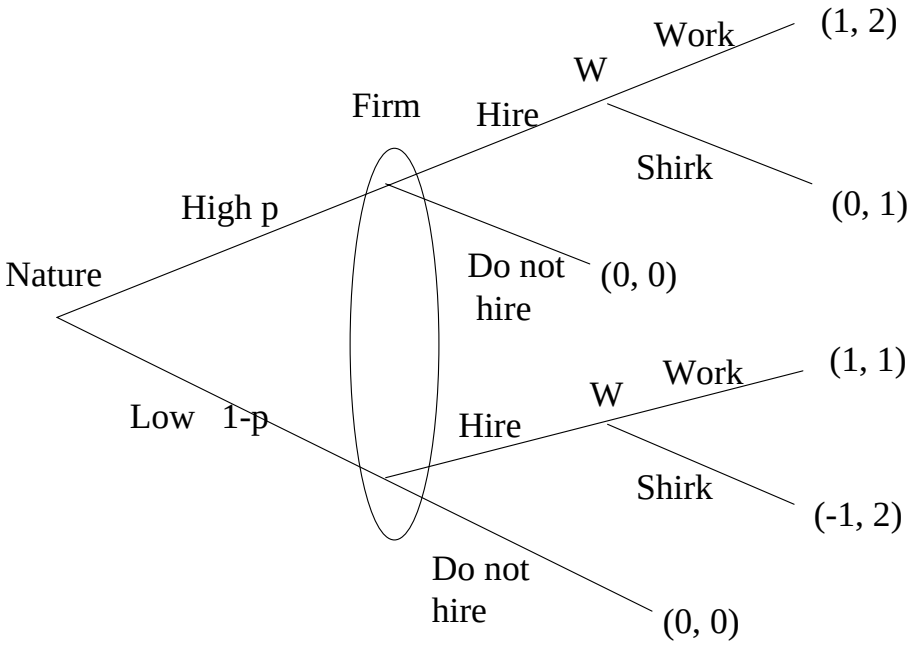
Muhamet Yildiz*

## 1 Static Games with Incomplete Information

An incomplete information game is a game where a party knows something that some other party does not know. For instance, a player may not know another player's utility function, while the player himself knows his utility function. Such situations are modeled through games where Nature moves and some players can distinguish certain moves of nature while some others cannot.

*Example*: Firm hiring a worker, the worker can be high or low ability and the firm does not know which.

*These notes are heavily based on the notes by Professor Daron Acemoglu, who taught this course in previous years.




In this game W knows whether he is of high (who will work) or low (who will shirk) ability worker, while the Firm does not know this; the Firm believes that the worker is of high ability with probability $p$. And all these are common knowledge. We call a player's private information as his "type". For instance, here W has two types – high and low – while Firm has only one type (as everything firm knows is common knowledge). Note that we represent the incomplete information game in an extensive form very similar to imperfect information games.

Formally, a static game with incomplete information is as follows. First, Nature chooses some $t = (t_1, t_2, \ldots, t_n) \in T$, where each $t \in T$ is selected with probability $p(t)$. Here, $t_i \in T_i$ is the type of player $i \in N = \{1, 2, \ldots, n\}$. Then, each player observes his own type, but not the others'. finally, players simultaneously choose their actions, each player knowing his own type. We write $a = (a_1, a_2, \ldots, a_2) \in A$ for any list of actions taken by all the players, where $a_i \in A_i$ is the action taken by player $i$. Such a static gamewith incomplete information is denoted by $(N, T, A, p)$.

Notice that a strategy of that player determines which action he will take at each information set of his, represented by some $t_i \in T_i$. That is, a strategy of a player $i$ is a function $s_i : T_i \to A_i$ –from his types to his actions. for instance, in example above, W has four strategies, such as (Work,Work) –meaning that he will work regardless nature chooses high or low– (Work, Shirk), (Shirk, Work), and (Shirk, Shirk).

Any Nash equilibrium of such a game is called Bayesian Nash equilibrium. For instance, for $p > 1/2$, a Bayesian Nash equilibrium of the game above is as follows. Worker chooses to

Work if he is of high ability, and chooses to Shirk if he is of low ability; and the firm Hires him.

Notice that there is another Nash equilibrium, where the worker chooses to Shirk regardless of his type and the firm doesn't hire him. Since the game has no proper subgame, the latter equilibrium is subgame perfect, even though it clearly relies on an "incredible" threat. this is a very common problem in games with incomplete information, motivating a more refined equilibrium concept we'll discuss in our next lectures.

It's very important to note that players' types may be "correlated", meaning that a player "updates" his beliefs about the other players' type when he learns his own type. Since he knows his type when he takes his action, he maximizes his expected utility with respect to the new beliefs he came to after "updating" his beliefs. We assume that he updates his beliefs using Bayes' Rule.

**Bayes' Rule** Let $A$ and $B$ be two events, then probability that $A$ occurs conditional on $B$ occurring is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

where $P(A \cap B)$ is the probability that $A$ and $B$ occur simultaneously and $P(B)$: the (unconditional) probability that $B$ occurs.

In static games of incomplete information, the application of Bayes' Rule will often be trivial, but as we move to study dynamic games of incomplete information, the importance of Bayes' Rule will increase.

Let $p_i(t'_{-i}|t_i)$ denote $i$'s belief that the types of all other players is $t'_{-i} = (t'_1, t'_2, ..., t'_{i-1}, t'_{i+1}, \ldots, t'_n)$ given that his type is $t_i$. [We may need to use Bayes' Rule if types across players are 'correlated'. But if they are independent, then life is simpler; players do not update their beliefs.]

With this formalism, a strategy profiles $s^* = (s^*_1, ..., s^*_n)$ is a Bayesian Nash Equilibrium in an $n$-person static game of incomplete information if and only if for each player $i$ and type $t_i \in T_i$,

$$s^*_i(t^1_i) \in \arg\max_{a_i} \sum u_i \left[s^*_i(t_i), ..., a_i, ..., s^*_N(t_N)\right] \times p_i(t'_{-i}|t_i)$$

where $u_i$ is the utility of player $i$ and $a_i$ denotes action. That is, for each player $i$ each possible type, the action chosen is *optimal* given the *conditional beliefs* of that type against the optimal strategies of all other players.

The remaining notes are about the applications and very sketchy; for the details see Gibbons.

**Example:** Cournot with Incomplete Information.

$P(Q) = a - Q$

$Q = q_1 + q_2$

$c_1(q_1) = cq_1$

Both firms Risk-Neutral

Firm 2's types (private information)

$c_2(q_2) = c_H q_2$ with probability $\theta$

$c_L q_2$ with probability $1 - \theta$

common knowledge among players.

How to find the Bayesian Nash Equilibrium?

Firm 2 has two possible types; and different actions will be chosen for the two different types.

$\{q_2(c_L), q_2(c_H)\}$

Suppose firm 2 is type high.

$\Longrightarrow$

$$\max_{q_2}(P - c_H)q_2 = [a - q_1 - q_2 - c_H]\, q_2$$

given the action of player $q_1$.

$$\Longrightarrow q_2(c_H) = \frac{a - q_1 - c_H}{2} \qquad (*)$$

Similarly suppose firm 2 is low type:

$$\max_{q_2}[a - q_1 - q_2 - c_H]\, q_2$$

$$q_2(c_L) = \frac{a - q_1 - c_H}{2} \qquad (**)$$

*Important Remark:* The same level of $q_1$ in both cases. Why??

Firm 1's problem

$$\max_{q_1}\theta\,[a - q_1 - q_2(c_H) - c]\, q_1 + (1 - \theta)\,[a - q_1 - q_2(c_L) - c]\, q_1$$

$$q_1 = \frac{\theta\,[a - q_2(c_H) - c] + (1 - \theta)\,[a - q_2(c_L) - c]}{2} \qquad (***)$$

Solve *, **, and *** for $q_1, q_2(c_L), q_2(c_H)$

$$q_2^*(c_H) = \frac{a - 2c_H + c}{3} + \frac{(1-\theta)(c_H - c_L)}{6}$$

$$q_2^*(c_L) = \frac{a - 2c_L + c}{3} + \frac{\theta(c_H - c_L)}{6}$$

$$q_1^* = \frac{a - 2c + \theta c_H + (1-\theta)c_L}{3}$$

**Harsaryi's Justification for Mixed Strategies**

| | O | F |
|---|---|---|
| O | $2 + t_1, 1$ | $0, 0$ |
| F | $0, 0$ | $1, 2 + t_2$ |

$t_1, t_2$ private information of players.

$t_1, t_2$ are independent draws from uniform distribution over $[0, X]$.

Harsayi shows that as $X \longrightarrow 0$ (as uncertainty disappears), we converge to a mixed strategy equilibrium where 1 plays 0 with probability 2/3 and 2 plays F with probability 2/3. See Gibbons for details.

**Auctions**

Two bidders for a unique good.

$v_i$ : valuation of bidder i.

Let us assume that $v_i$'s are drawn independently from a uniform distribution over $[0, 1]$. $v_i$ is player i's private information. The game takes the form of both bidders submitting a bid, then the highest bidder wins and pays her bid.

Let $b_i$ be player i's bid.

$v_i(b_1, b_2, v_1, v_2) = v_i - b_i$ if $b_i > b_j$

$\frac{v_i - b_i}{2}$ if $b_i = b_j$

$0$ if $b_i < b_j$

$$\max_{b_i}(v_i - b_i)Prob\{b_i > b_j(v_j)|\text{given beliefs of player i}) + \frac{1}{2}(v_i - b_i)Prob\{b_i = b_j(v_j)|...)$$

$\frac{1}{2}(v_i - b_i)Prob\{b_i = b_j(v_j)|...) = 0$ since a continuum of possibilities.

Let us first conjecture the form of the equilibrium: Conjecture: Symmetric and linear equilibrium

$$b = a + cv.$$

Then

$$\max_{b_i}(v_i - b_i)Prob\{b_i \geq a + cv_j\} =$$
$$(v_i - b_i)Prob\{v_j \leq \frac{b_i - a}{c}\} = (v_i - b_i) \cdot \frac{(b_i - a)}{c}$$

FOC:

$$b_i = \frac{v_i + a}{2} \qquad \text{if} \quad v_i \geq a$$
$$= a \qquad \text{if} \quad v_i < a \tag{1}$$

A linear strategy is BR to a linear strategy only if $a = 0$

$$\Longrightarrow b_i = \frac{1}{2}v_i$$

$$b_i = \frac{1}{2}v_j$$

Double Auction

Seller names $P_s$

Buyer names $P_b$

$P_b < P_s$ no trade

$P_b \geq P_s$ trade at $p = \frac{P_b + P_s}{2}$

Valuations again private information.

$V_b$ uniform over $(0, 1)$

$V_s$ uniform over $(0, 1)$ and independent from $V_b$

Strategies $P_b(V_b)$ $P_s(V_s)$

The buyer maximizes

$$\max_{P_b}\left[V_b - \frac{P_b + E\{P_s(V_s)|P_b \geq P_s(V_s)\}}{2}\right] \times Prob\{P_b \geq P_s(V_s)\}$$

where $E(P_s(V_s)|P_b \geq P_s(V_s)$ expected seller bid *conditional* on $P_b$ being greater than $P_s(V_s)$.

Similarly, the seller maximizes

$$\max_{P_s} [P_s + E\{P_b(V_b)|P_b(V_b) \geq P_s] - V_s] \times Prob\{P_b(V_b) \geq P_s\}$$

Equilibrium $P_s(V_j)BR$ to $P_b(V_b)$

$P_b(V_b)BR$ to $P_s(V_s)$

Bayesian Nash Equilibria?

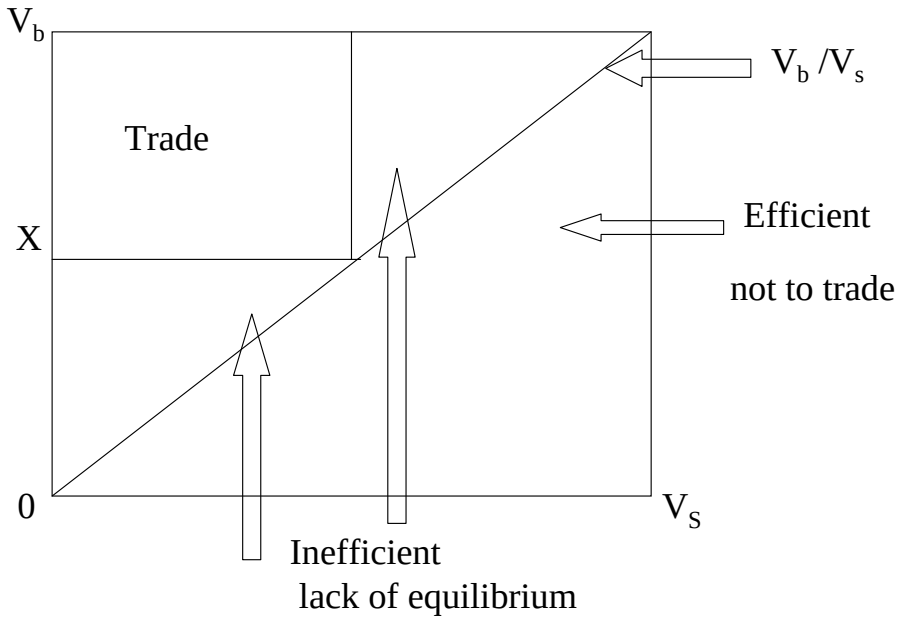
There are many: Let us construct some examples

1. Seller $P_s = X$ if $V_s \leq X$

$P_b = X$ if $V_b \geq X$

An equilibrium with "fixed" price.

Why is this an equilibrium? because given $P_s = X$ if $V_s \leq X$, the buyer does not want to trade with $V_b < X$ and with $V_b > X, P_b = X$ is optimal.




See Gibbons for other equilibria.

*Theorem (*Myerson and Satterthwrite): There is no Bayesian Nash Equilibrium without inefficient lack of trade.

# 14.12 Game Theory Lecture Notes
# Lectures 16-18

Muhamet Yildiz

## 1 Dynamic Games with Incomplete Information

In these lectures, we analyze the issues arise in a dynamics context in the presence of incomplete information, such as how agents should interpret the actions the other parties take. We define perfect Bayesian Nash equilibrium, and apply it in a sequential bargaining model with incomplete information. As in the games with complete information, now we will use a stronger notion of rationality — sequential rationality.

## 2 Perfect Bayesian Nash Equilibrium

Recall that in games with complete information some Nash equilibria may be based on the assumption that some players will act sequentially irrationally at certain information sets off the path of equilibrium. In those games we ignored these equilibria by focusing on subgame perfect equilibria; in the latter equilibria each agent's action is sequentially rational at each information set. Now, we extend this notion to the games with incomplete information. In these games, once again, some Bayesian Nash equilibria are based on sequentially irrational moves off the path of equilibrium.

Consider the game in Figure 1. In this game, a firm is to decide whether to hire a worker, who can be hard-working (High) or lazy (Low). Under the current contract, if the worker is hard-working, then working is better for the worker, and the firm makes profit of 1 if the worker works. If the worker's lazy, then shirking is better for him, and the firm will lose 1 if the worker shirks. If the worker is sequentially rational, then he will work if he's hard-working and shirk if he's lazy. Since the firm finds the worker

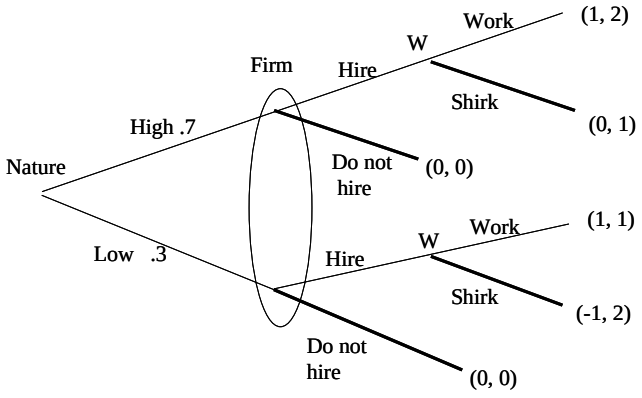



Figure 1:

more likely to be hard-working, the firm will hire the worker. But there is another Bayesian Nash equilibrium: the worker always shirks (independent of his type), and therefore the firm does not hire the worker. This equilibrium is indicated in the figure by the bold lines. It is based on the assumption that the worker will shirk when he is hard-working, which is sequentially irrational. Since this happens off the path of equilibrium, such irrationality is ignored in the Bayesian Nash equilibrium—as in the ordinary Nash equilibrium.

We'll now require sequential rationality at each information set. Such equilibria will be called perfect Bayesian Nash equilibrium. The official definition requires more details.

For each information set, we must specify the beliefs of the agent who moves at that information set. Beliefs of an agent at a given information set are represented by a probability distribution on the information set. In the game in figure 1, the players' beliefs are already specified. Consider the game in figure 2. In this game we need to specify the beliefs of player 2 at the information set that he moves. In the figure, his beliefs are summarized by $\mu$, which is the probability that he assigns to the event that player 1 played T given that 2 is asked to move.

Given a player's beliefs, we can define sequential rationality:

**Definition 1** *A player is said to be* sequentially rational *iff, at each information set he is to move, he maximizes his expected utility given his beliefs at the information set (and given that he is at the information set) – even if this information set is precluded by his*

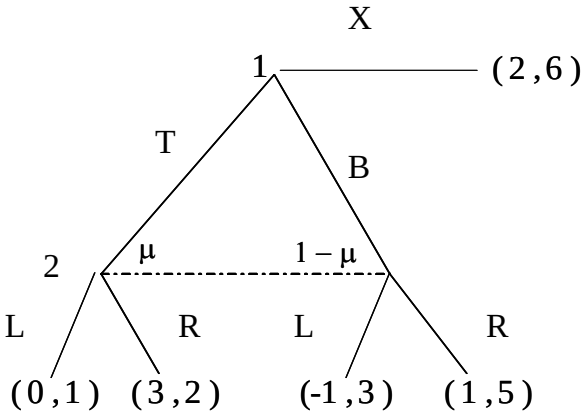



Figure 2:

*own strategy.*

In the game of figure 1, sequential rationality requires that the worker works if he is hard-working and shirks if he is lazy. Likewise, in the game of figure 2, sequential rationality requires that player 2 plays R.

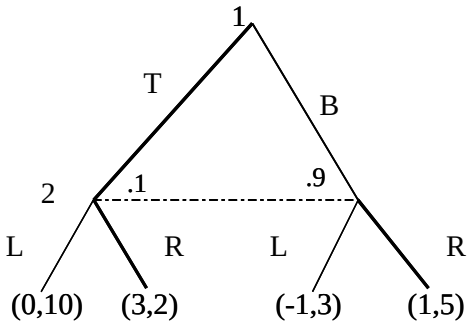



Figure 3:

Now consider the game in figure 3. In this figure, we depict a situation in which player 1 plays T while player 2 plays R, which is not rationalizable. Player 2 assigns probability .9 to the event that player 1 plays B. Given his beliefs, player 2's move is sequentially rational. Player 1 plays his dominant strategy, therefore his move is sequentially rational. The problem with this situation is that player 2's beliefs are not

consistent with player 1's strategy. In contrast, in an equilibrium a player maximizes his expected payoff given the other players' strategies. Now, we'll define a concept of consistency, which will be required in a perfect Bayesian Nash equilibrium.

**Definition 2** *Given any (possibly mixed) strategy profile s, an information set is said to be* on the path of play *iff the information set is reached with positive probability according to s.*

**Definition 3** ***(Consistency on the path)*** *Given any strategy profile s and any information set I on the path of play of s, a player's beliefs at I is said to be consistent with s iff the beliefs are derived using the Bayes' rule and s.*

For example, in figure 3, consistency requires that player 2 assigns probability 1 to the event that player 1 plays T. This definition does not apply off the equilibrium path. Consider the game in Figure 4. In this game, after player 1 plays E, there

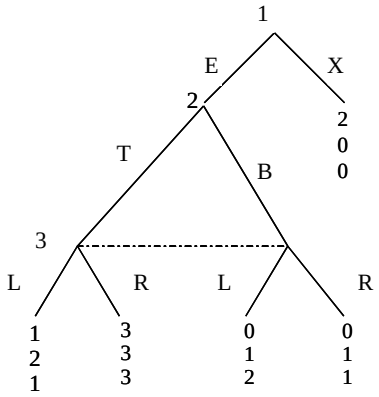



Figure 4:

is a subgame with a unique rationalizable strategy profile: 2 plays T and 3 plays R. Anticipating this, player 1 must play E. Now consider the strategy profile (X,T,L), in which player 1 plays X, 2 plays T, and 3 plays L, and assume that, at his information set, player 3 assigns probability 1 to the event that 2 plays B. Players' moves are all sequentially rational, but player 3's beliefs are not consistent with what the other players play. Since our definition was valid only for the information sets that are on the path of equilibrium, we could not preclude such beliefs. Now, we need to extend our definition

of consistency to the information sets that are off the path of equilibrium. The difficulty is that the information sets off the path of equilibrium are reached with probability 0 by definition. Hence, we cannot apply Bayes' formula to compute the beliefs. To check the consistency we might make the players "tremble" a little bit so that every information sets is reached with positive probability. We can then apply Bayes rule to compute the conditional probabilities for such a perturbed strategy profile. Consistency requires that the players' beliefs must be close to the probabilities that are derived using Bayes' rule for some such small tremble (as the size of the tremble goes to 0). In figure 4, for any small tremble (for player 1 and 2), the Bayes rule yields a probability close to 1 for the event that player 2 plays T. In that case, consistency requires that player 3 assigns probability 1 to this event. **Consistency is required both on and off the equilibrium path.**

In the definition of sequential rationality above, the players' beliefs about the nodes of the information set are given but his beliefs about the other players' play in the continuation game are not specified. In order to have an equilibrium, we also need these beliefs to be specified consistently with the other players' strategies.

**Definition 4** *A* ***strategy profile*** *is said to be* ***sequentially rational*** *iff, at each information set, the player who is to move maximizes his expected utility given*

1. *his beliefs at the information set, and*
2. *given that the other players play according to the strategy profile in the continuation game (and given that he is at the information set).*

**Definition 5** *A* Perfect Bayesian Nash Equilibrium *is a pair (s,b) of strategy profile and a set of beliefs such that*

1. *s is sequentially rational given beliefs b, and*
2. *b is consistent with s.*

The only perfect Bayesian equilibrium in figure 4 is (E,T,R). This is the only subgame perfect equilibrium. Note that every perfect Bayesian equilibrium is subgame perfect.

# 3 Examples

**Beer-Quiche Game** Consider the game in figure 5. In this game, player one has two types: weak or strong. Player 2 thinks that player 1 is strong with probability .9. Player 2, who happens to be a bully, wants to fight with player 1 if player 1 is weak and would like to avoid a fight if player 1 is strong. Player 1 is about to order his breakfast, knowing that player 2 observes what player 1 orders. He prefers beer if he is strong, and he prefers quiche if he is weak. He wants to avoid a fight.

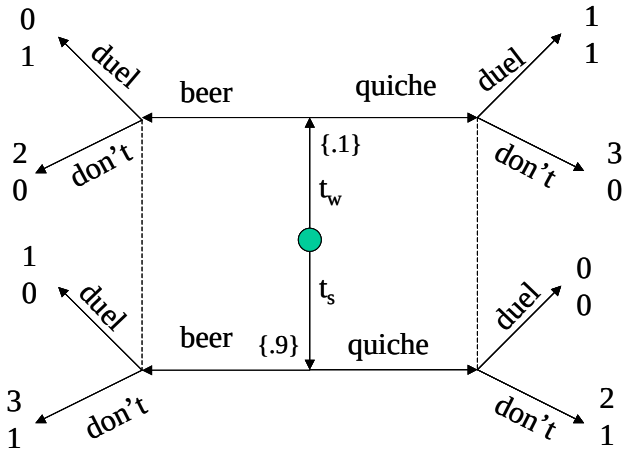



Figure 5:

This game has two equilibria. (For each equilibrium there is a continuum of mixed strategy equilibria off the path of equilibrium.) First, consider the perfect Bayesian Nash equilibrium depicted in figure 6. We need to check two things: sequential rationality and consistency. Let us first check that the strategy profile is sequentially rational. In his information set on the right, player 2 is sure that player 1 is weak, hence he chooses to duel. When he sees that player 1 is having beer for his breakfast he assigns probability .9 to the event that player 1 is strong. Hence, his expected payoffs from duel is $.9 \times 1 = .9$, and his expected payoff is .1 otherwise. Therefore, his moves are sequentially rational. Now consider the strong type of player 1. If he chooses beer, then he gets 3, and if he chooses quiche, then he gets 0. He chooses beer. Now consider the weak type. If he chooses beer, he gets 2, while he gets only 1 if he chooses quiche. He chooses beer. Therefore, player 1's moves are sequentially rational.

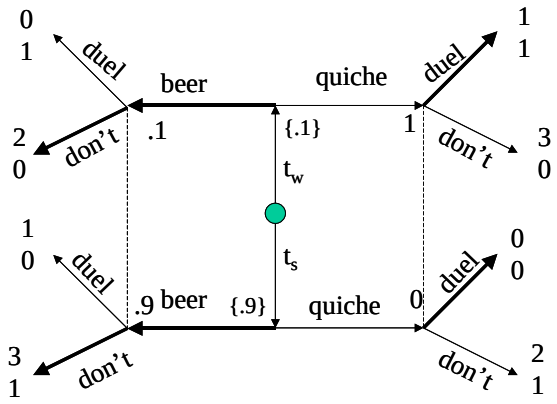



Figure 6:

Let's now check the consistency. The information set after the beer is on the path of equilibrium; hence we need to use the Bayes' rule. The probabilities .9 and .1 are indeed computed through Bayes' rule. The information set after the quiche is off the equilibrium path. In this game, any belief off the equilibrium path is consistent. For the present belief, which puts probability 1 to the weak type, consider a perturbation in which player 1 trembles and orders quiche with probability $\varepsilon$ if he is weak, and he does not tremble if his strong. Now Bayes' rule yields $\varepsilon/\varepsilon = 1$ as the conditional probability of being weak given quiche. Therefore, the players beliefs are consistent, and we have a perfect Bayesian Nash equilibrium.

Note that we have a continuum of equilibria in which player 1 orders beer. After the quiche, player 2 assigns equal probabilities to each node and mixes between duel and not duel, where the probability of duel is at least .5. Check also that there is a perfect Bayesian Nash equilibrium in which player 1 orders quiche independent of this type, and player 2 fights when he observes a beer.

**Another example** Consider the game in figure 7. sequential rationality requires that at the last note in the upper branch player 1 goes down, and at the last node of the lower branch player 1 goes across. Moreover, it requires that player 1 goes across at the first node of the lower branch. Therefore, player 1 must go across throughout the lower branch and go down at the last node of the upper branch at any perfect Bayesian Nash equilibrium. We now show that, in any perfect Bayesian Nash equilibrium, the players

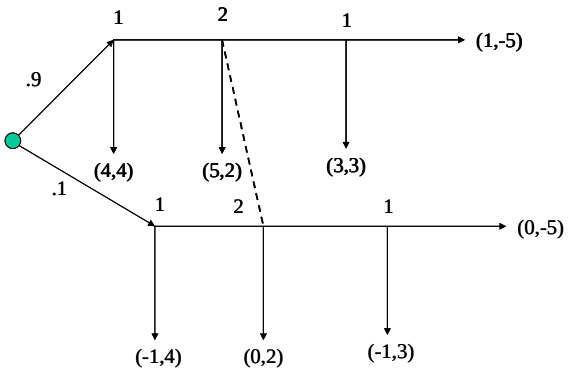



Figure 7:

must play mixed strategies at the remaining information sets (i.e., at the first node of the upper branch, and at the information set of player 2). Suppose that player 1 goes down with probability 1 at the first node on the upper branch. Then, by Bayes' rule, player 2 must assign probability 1 to the lower branch at his information set and must go down with probability 1. In that case, it is better for player 1 to go across and get 5, rather than going down and getting 4 —a contradiction. Therefore, player 1 must go across with positive probability at the first node of a upper branch. Now, suppose that player 1 goes across with probability 1 at this node. Then by Bayes' rule, player 2 must assign probability .9 to the upper branch in his information set. If he goes down, he gets 2; if he goes across, he gets $.9 \times 3 + .1 \times (-5) = 2.2$. Then, he must go across with probability 1. In that case, player 1 must go down with probability 1 at the node at hand—another contradiction. Therefore, player 1 must mix at the present node. In order to have this, player 1 must be indifferent between going across and going down. Let's write $\beta$ for the probability that 2 goes across. For indifference, we must have

$$4 = 5\left(1-\beta\right) + 3\beta = 5 - 2\beta,$$

i.e.,

$$\beta = 1/2.$$

Player 2 must also play a mixed strategy. Since player 2 plays a mixed strategy, he must be indifferent. Let's write $\mu$ for the probability he assigns to the upper branch at his information set. For indifference, we must have

$$2 = 3\mu + \left(1-\mu\right)\left(-5\right) = 8\mu - 5,$$

i.e.,

$$\mu = 7/8.$$

If player 1 goes across with probability $\alpha$, then by Bayes' rule, we must have

$$\mu = \frac{.9\alpha}{.9\alpha + .1} = \frac{7}{8},$$

hence

$$\alpha = 7/9.$$

Therefore, there is a unique perfect Bayesian Nash equilibrium as depicted in figure 8.

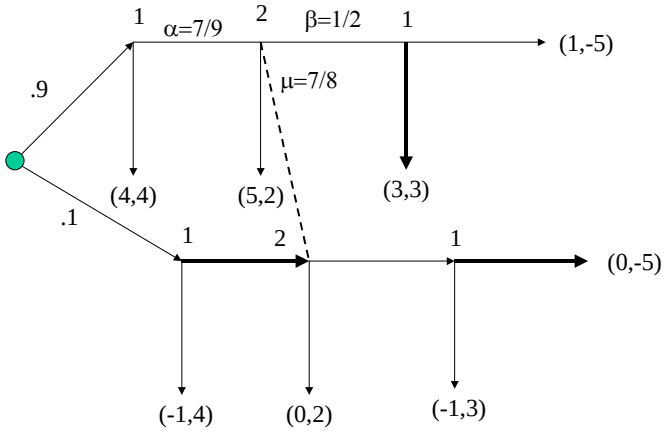



Figure 8:

# 4 Sequential bargaining

## 4.1 A one-period model with two types

We have a seller S with valuation 0 and a buyer B with valuation $v$. B knows $v$, S does not; S believes that $v = 2$ with probability $\pi$, and $v = 1$ with probability $1 - \pi$. We have the following moves. First, S sets a price $p \geq 0$. Knowing $p$, B either buys, yielding $(p, v - p)$ (where the first entry is the payoff of the seller), or does not, yielding (0,0). The game is depicted in Figure 9.

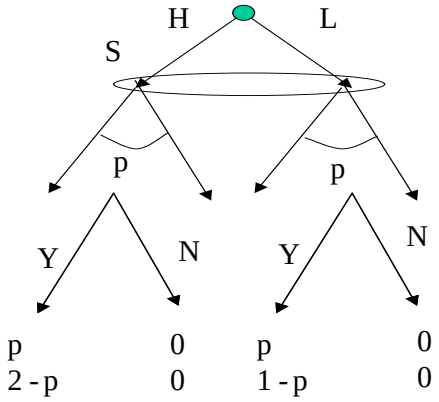



Figure 9:

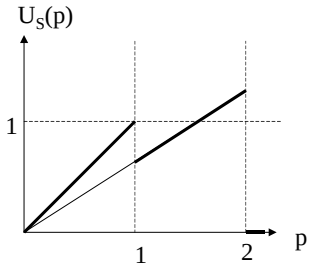



Figure 10:

The perfect Bayesian Nash equilibrium is as follows. B buys iff $v \geq p$. If $p \leq 1$, both types buy, and S gets $p$. If $1 < p \leq 2$, only H-type buys, and S gets $\pi p$. If $p > 2$, no one buys. The expected payoff of S is plotted in Figure 10. S offers 1 if $\pi < \frac{1}{2}$, and he offers 2 if $\pi > \frac{1}{2}$. He is indifferent between the prices 1 and 2 when $\pi = 1/2$.

## 4.2 A two-period model with two types

Consider the same buyer and the seller, but allow them to trade at two dates $t = 0, 1$. The moves are as follows. At $t = 0$, S sets a price $p_0 \geq 0$. B either buys, yielding $(p_0, v - p_0)$, or does not. If he does not buy, then at $t = 1$, S sets another price $p_1 \geq 0$; B either buys, yielding $(\delta p_1, \delta(v - p_1))$, or does not, yielding (0,0).

The equilibrium behavior at $t = 1$ is the same as above. Let's write

$$\mu = \Pr(v = 2 | \text{history at } t = 1).$$

B buys iff $v \geq p_1$. If $\mu > \frac{1}{2}$, $p_1 = 2$; if $\mu < \frac{1}{2}$, $p_1 = 1$. If $\mu = \frac{1}{2}$, S is indifferent between 1 and 2.

Given this, B with $v = 1$ buys at $t = 0$ if $p_0 \leq 1$. Hence, by Bayes' rule, if $p_0 > 1$,

$$\mu = \Pr(v = 2 | p_0,\ t = 1) \leq \pi.$$

**When** $\pi < 1/2$, this determines the equilibrium. This is because

$$\mu = \Pr(v = 2 | p_0,\ t = 1) \leq \pi < 1/2,$$

and thus

$$p_1 = 1.$$

Hence, B with $v = 2$ buys at $t = 0$ if

$$(2 - p_0) \geq \delta(2 - 1) = \delta.$$

This is true iff

$$p_0 \leq 2 - \delta.$$

Now S has two options: either set $p_0 = 1$ and sell the good with probability 1, yielding payoff 1, or set $p_0 = 2 - \delta$, and sell to the high-value buyer at $t = 0$ and sell the low-value buyer at $t = 1$. The former is better, and thus $p_0 = 1$:

$$\pi (2 - \delta) + (1 - \pi) \delta = 2\pi (1 - \delta) + \delta < (1 - \delta) + \delta = 1.$$

**Consider the case** $\pi > 1/2$**.** In that case, after any price $p_0 \in (2 - \delta, 2)$, the players must mix (see the slides). At any $p_0 > 2 - \delta$, since B mixes at $t = 1$, we must have

$$\mu (p_0) = \Pr(v = 2 | p_0,\ t = 1) = 1/2.$$

Write $\beta (p_0)$ for the probability that high-value buyer does not buy at price $p_0$. Then, by Bayes' rule,

$$\mu (p_0) = \frac{\beta (p_0) \pi}{\beta (p_0) \pi + (1 - \pi)} = \frac{1}{2},$$

i.e.,

$$\beta (p_0) = (1 - \pi) / \pi.$$

Since the buyer with $v = 2$ mixes (i.e., $\beta(p_0) \in (0,1)$), he must be indifferent towards buying at $p_0$. That is, writing $\gamma(p_0) = \Pr(p_1 = 1|p_0)$, we have

$$2 - p_0 = \delta\gamma(p_0)$$

i.e.,

$$\gamma(p_0) = (2 - p_0)/\delta.$$

## 4.3 A one-period model with continuum of types

Modify the one-period model above by letting $v$ be distributed uniformly on some interval $[0, a]$. In equilibrium, again B buys at price $p$ iff $v \geq p$. S gets

$$U(p) = p\Pr(v \geq p) = p(a - p)/a.$$

Therefore, S sets

$$p = a/2.$$

## 4.4 A two-period model with continuum of types

Modify the two-period model above by letting $v$ be distributed uniformly on $[0, 1]$. B buys at $p_0$ iff

$$v - p_0 \geq \delta(v - E[p_1|p_0]), \tag{1}$$

where $E[p_1|p_0]$ is the expected value of $p_1$ given $p_0$. This inequality holds iff

$$v \geq \frac{p_0 - E[p_1|p_0]}{1 - \delta} \equiv a(p_0).$$

Hence, if B does not buy at price $p_0$, S's posterior belief will be that $v$ is uniformly distributed on $[0, a(p_0)]$, in which case he will set the price at $t = 1$ to

$$p_1(p_0) = a(p_0)/2$$

as shown above. Substituting this into the previous definition we obtain

$$a(p_0) = \frac{p_0 - \delta E[p_1|p_0]}{1 - \delta} = \frac{p_0 - \delta a(p_0)/2}{1 - \delta},$$

i.e.,

$$a(p_0) = \frac{p_0}{1 - \delta/2}.$$

(One could obtain this, simply by observing that (1) is an equality when $v = a(p_0)$ and $E[p_1|p_0] = a(p_0)/2$.) Notice that, if S offers $p_0$, in equilibrium, he sells to the types $v \geq a(p_0)$ at price $p_0$ (at date $t = 0$), to the types with $a(p_0)/2 \leq v \leq a(p_0)$ at $p_1 = a(p_0)/2$ at date 1, and does not sell to the types $v < a(p_0)/2$ at all. His expected payoff is

$$\begin{aligned} U_S(p_0) &= \Pr(v > a(p_0))\, p_0 + \delta \Pr(p_1 \leq v < a(p_0))\, p_1 \\ &= (1 - a(p_0))\, p_0 + \delta (a(p_0)/2)(a(p_0)/2) \\ &= \left(1 - \frac{p_0}{1 - \delta/2}\right) p_0 + \delta \left(\frac{p_0}{2 - \delta}\right)^2. \end{aligned}$$

The first order condition yields

$$0 = U_S'(p_0) = 1 - \frac{2p_0}{1 - \delta/2} + \frac{2\delta p_0}{(2 - \delta)^2},$$

i.e.,

$$p_0 = \frac{(1 - \delta/2)^2}{2(1 - 3\delta/4)}.$$

# 14.12 Game Theory Lecture Notes
# Reputation and Signaling

Muhamet Yildiz

In these notes, we discuss the issues of reputation from an incomplete information point of view, using the centipede game. We also introduce the signaling games and illustrate the separating, pooling, and partial-pooling equilibria.

## 1 Reputation

Consider a game in which a player $i$ has two types, say A and B. Imagine that if the other players believe that $i$ is of type A, then $i$'s equilibrium payoff will be much higher than his equilibrium payoff when the other players believe that he is of type B. If there's a long future in the game and $i$ is patient, then he will act as if he is of type A even when his type is B, in order to convince the other players that he is of type A. In other words, he will try to form a reputation for being of type A. This will change the equilibrium behavior dramatically when the other players assign positive probability to each type. For example, if a seller thinks that the buyer does not value the good that much, then he will be willing to sell the good at a low price. Then, even if the buyer values the good highly, he will pretend that he does not value the good that much and will not buy the object at higher prices—although he could have bought at those prices if it were common knowledge that he values the object highly. (If the players are sufficiently patient, then in equilibrium the price will be very low.) Likewise, in the entry deterrence game, if it is possible that the incumbent gains from a fight in case of an entry, if this is the incumbent's private information, and if there is a long future in the game, then he will fight whenever the entrant enters, in order to form a reputation for being a fighter and deter the future entries. In that case, the entrants will avoid

entering even if they are confident that the incumbent is not a fighter. We will now illustrate this notion of reputation formation on the centipede game.

Consider the centipede game in figure 1. In this game, a player prefers to exit (or

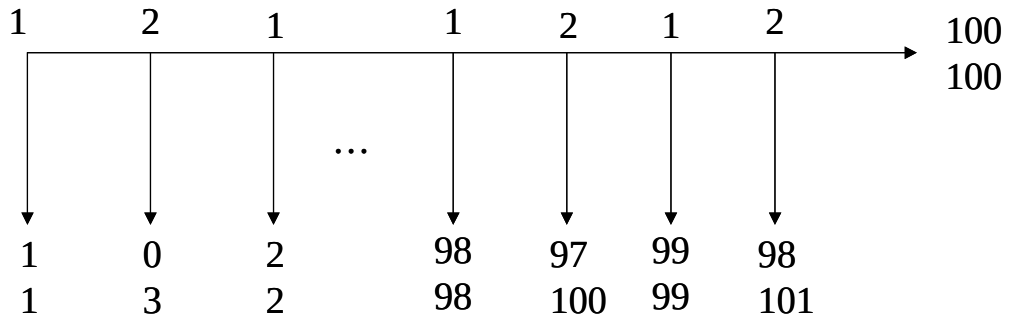



Figure 1:

to go down) if he believes that the other player will exit at the next node. Moreover, player 2 prefers exiting at the last node. Therefore, the unique backward induction outcome in this game is that each player goes down at each node. In particular, player 1 goes down at the first node and the game ends. This outcome is considered to be very counterintuitive, as the players forego very high payoffs. We will see that it is not robust to asymmetric information, in the sense that the outcome would change dramatically if there were a slight probability that a player is of a certain "irrational" type. In figure 2, we consider such a case. Here, player 2 assigns probability .999 to the event that player 1 is a regular rational type, but she also assigns probability .001 to the event that player 1 is a "nice" irrational type who would not want to exit the game. We index the nodes by $n$ starting from the end. Sequential rationality requires that player 1 goes across at each information set on the lower branch. Moreover at the last information set ($n = 1$), player 2 goes down with probability 1. These facts are indicated in figure 3. We will now prove further facts. We need some notation. Let's write $\mu_n$ for the probability player 2 assigns to the lower branch at information set $n$. Moreover, let $p_n$ be the probability that player 1 goes down at node $n$ if he is rational.

**Facts about the perfect Bayesian Nash equilibrium**

1. *For any $n > 1$, player 2 goes across with positive probability.* Suppose that player

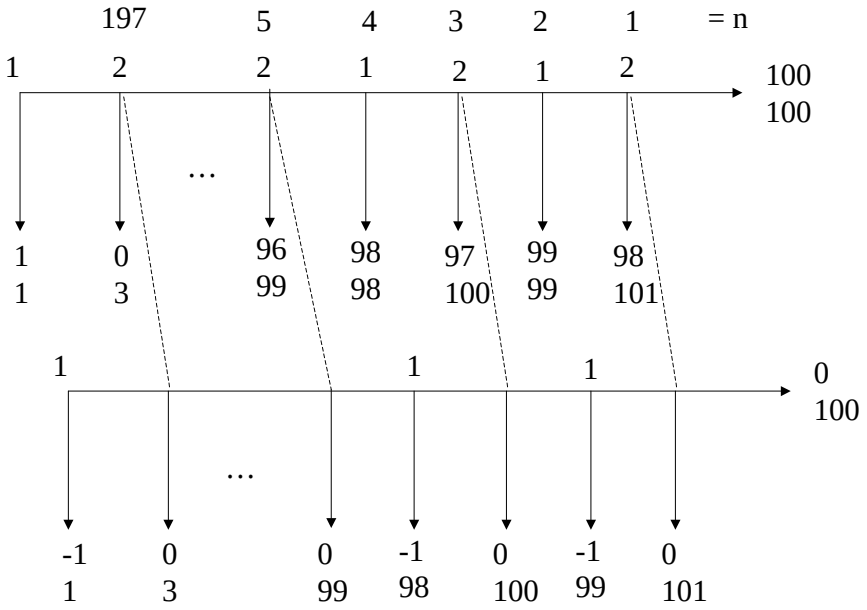



Figure 2:

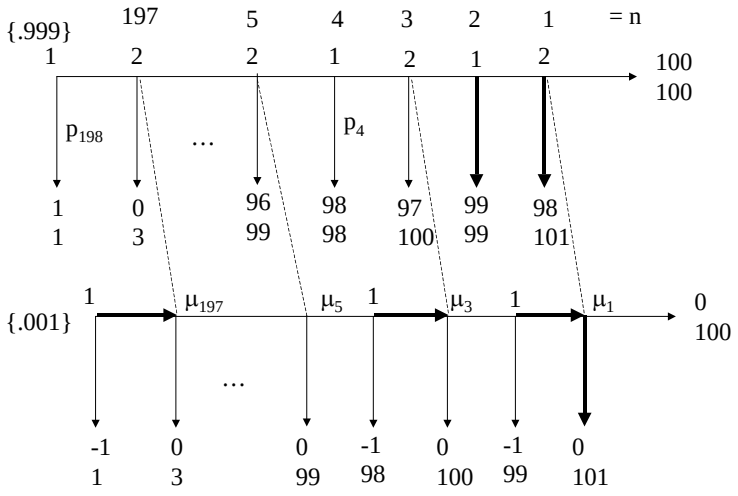



Figure 3:

2 goes down with probability 1 at $n > 1$. Then, if rational, player 1 must go down with probability 1 at $n + 1$. Hence, we must have $\mu_n = 1$. That is, player 2 is sure that player 1 is irrational and will go across until the end. She must then go across with probability 1 at $n$.

2. *Every information set of player 2 is reached.* (This is because irrational player 1 and player 2 go across with positive probability.) Hence, the beliefs are determined by the Bayes' rule.

3. *For any $n > 2$, rational player 1 goes across with positive probability.* Suppose that rational player 1 goes down with probability 1 at $n > 2$. Then, $\mu_{n-1} = 1$, and thus player 2 must go across with probability 1 at $n-1$. Then, rational player 1 must go across at $n$ with probability 1.

4. *If player 2 strictly prefers to go across at n, then*

   (a) *1 goes across with probability 1 at n+1,*

   (b) *2 must strictly prefer to go across at n+2,*

   (c) *2's posterior at n is her prior.*

   Parts a and b must be clear by now. Inductive application of parts a and b imply that players go across with probability 1 until $n$. Then, c follows from the Bayes' rule.

5. *If rational player 1 goes across with probability 1 at n, then 2's posterior at $n-1$ is her prior.* (This follows from 4.c.)

6. *There exists $n^*$ such that each player go across with probability 1 before $n^*$ (i.e., when $n > n^*$), and rational player 1 and player 2 mix after $n^*$ (i.e., when $n < n^*$).* (This fallows from 1,3, and 4.)

We will now compute the equilibrium. Firstly, note that player 2 mixes at $n = 3$ (i.e., $n^* > 3$). [Otherwise, we would have $\mu_3 = .001$ by 4.c, when player 2 would go down with probability 1 at $n = 3$, contradicting Fact 1.] That is, she is indifferent between going across and going down at $n = 3$. Hence,

$$100 = 101\mu_3 + 99\,(1 - \mu_3) = 99 + 2\mu_3,$$

i.e.,

$$\mu_3 = 1/2.$$

Now consider any odd $n$ with $3 < n < n^*$; player 2 moves at $n$. Write $x$ for the payoff of player 2 at $n$ (if she goes down). Since player 2 is indifferent being going across and going down, we have

$$x = \mu_n(x+1) + (1-\mu_n)[(x-1)p_{n-1} + (1-p_{n-1})(x+1)].$$

The left hand side of this equation is the payoff if she goes down. Let us look at the right hand side. If she goes across, with probability $\mu_n$, player 1 is irrational and will go across at $n-1$ with probability 1, reaching the information set at $n-2$. Since player 2 is indifferent between going down and going across at the information set $n-2$, the expected payoff from reaching this information set for player 2 is $x+1$. This gives the first term. With probability $1-\mu_n$, he is rational and will go down at $n-1$ with probability $p_{n-1}$, yielding payoff $x-1$ for player 2, and will go across with probability $1-p_{n-1}$, reaching the information set at $n-2$, which yields $x+1$ for player 2. This gives the second term. After some algebraic manipulations, this equation simplifies to

$$(1-\mu_n)p_{n-1} = 1/2. \tag{1}$$

But, by the Bayes' rule, we have

$$\begin{aligned} \mu_{n-2} &= \frac{\mu_n}{\mu_n + (1-\mu_n)(1-p_{n-1})} \\ &= \frac{\mu_n}{\mu_n + 1 - \mu_n - (1-\mu_n)p_{n-1}} = \frac{\mu_n}{1-(1-\mu_n)p_{n-1}} \\ &= 2\mu_n, \end{aligned} \tag{2}$$

where the last equality is due to (1). Therefore,

$$\mu_n = \frac{\mu_{n-2}}{2}.$$

By applying the last equality iteratively, we obtain

$$\begin{aligned}
\mu_3 &= 1/2 \\
\mu_5 &= \mu_3/2 = 1/4 \\
\mu_7 &= \mu_5/2 = 1/8 \\
\mu_9 &= \mu_7/2 = 1/16 \\
\mu_{11} &= \mu_9/2 = 1/32 \\
\mu_{13} &= \mu_{11}/2 = 1/64 \\
\mu_{15} &= \mu_{13}/2 = 1/128 \\
\mu_{17} &= \mu_{15}/2 = 1/256 \\
\mu_{19} &= \mu_{17}/2 = 1/512 \\
\mu_{21} &= \mu_{19}/2 = 1/1024 < .001.
\end{aligned}$$

Therefore, $n^* = 20$. At any even $n < n^*$, player 1 goes across with probability

$$p_n = \frac{1}{2(1-\mu_{n+1})},$$

and at $n-1$ player 2 mixes so that player 1 is indifferent between going across and going down. If we write $q_{n-1}$ for the probability that 2 goes down at $n-1$ and $y$ for the payoff of rational 1 at $n$, then we have

$$y = q_{n-1}(y-1) + (1-q_{n-1})(y+1) = y + 1 - 2q_{n-1},$$

i.e.,

$$q_{n-1} = 1/2.$$

At $n^*+1 = 21$, $\mu_{n^*+1} = .001$, and at $n^*-1 = 19$, $\mu_{n^*-1} = 1/512$. Hence, by Bayes' rule,

$$1/512 = \mu_{n^*-1} = \frac{\mu_{n^*+1}}{1-\left(1-\mu_{n^*+1}\right)p_{n^*}} = \frac{.001}{1-.999p_{n^*}},$$

yielding

$$p_{n^*} = \frac{1-.512}{.999} \cong .488.$$

At any $n > n^*$, each player goes across with probability 1.

# 2 Signaling Games

In a signaling game, there are two players: Sender (denoted by S) and Receiver (denoted by R). First, Nature selects a type $t_i$ from a type space $T = \{t_1, \ldots, t_I\}$ with probability $p(t_i)$. Sender observes $t_i$, and then chooses a message $m_j$ from a message space $M = \{m_1, \ldots, m_J\}$ to be sent to the receiver. Receiver observes $m_j$ (but not $t_i$), and then chooses an action $a_k$ from an action space $A = \{a_1, \ldots, a_K\}$. The payoffs for the Sender and the Receiver are $U_S(t_i, m_j, a_k)$ and $U_R(t_i, m_j, a_k)$, respectively. A good example for a signaling game is the Beer-Quiche game (depicted in figure 4).

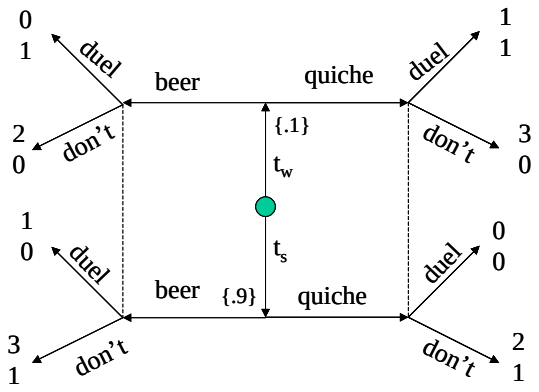



Figure 4:

In this game, Sender is player 1; Receiver is player 2. Type space is $T = \{t_s, t_w\}$, messages are beer and quiche (i.e., $M = \{\text{beer, quiche}\}$), and the actions are "duel" and "don't". In this game there were two equilibrium outcomes; either both types have beer, or both types have quiche. (The former equilibrium is depicted in figure 5.) Such equilibria are called pooling equilibrium, because the types are all send the same message, and thus no information is conveyed to the receiver; the receiver's posterior beliefs on the path of equilibrium is the same as his priors. Formally,

**Definition 1** *A* pooling equilibrium *is an equilibrium in which all types of sender send the same message.*

Another type of perfect Bayesian Nash equilibrium is separating equilibrium:

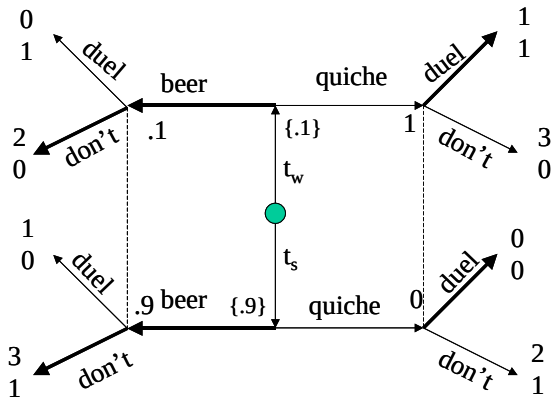



Figure 5:

**Definition 2** *A* separating equilibrium *is an equilibrium in which all types of sender send different messages.*

Since each type sends a different message on the path of a separating equilibrium receiver learns the sender's type (the truth). Such an equilibrium is depicted in Figure 6. Notice that player 2 assigns probability 1 to the strong type after beer and to the weak type after quiche.

Most equilibria in many games will be partially pooling and partially separating:

**Definition 3** *A partially separating/pooling equilibrium is an equilibrium in which some types of sender send the same message, while some others sends some other messages.*

Such an equilibrium is depicted in figure 7 for a beer-quiche game in which player 1 is very likely to be weak. In this equilibrium, strong player 1 orders beer, while the weak type mixes between beer and quiche so that after beer, player 2 finds the types equally likely, making him indifferent between the duel and not duel. After beer, player 2 mixes between duel and not duel with equal probabilities so that the weak type of player 1 is indifferent between beer and quiche, allowing him to mix. Note that the beliefs are derived via the Bayes' rule. Note also that, after quiche, player 2 knows that player 1 is weak, while he is not sure about player 1's type after beer — although he updates his belief and find player 1 more likely to be strong.

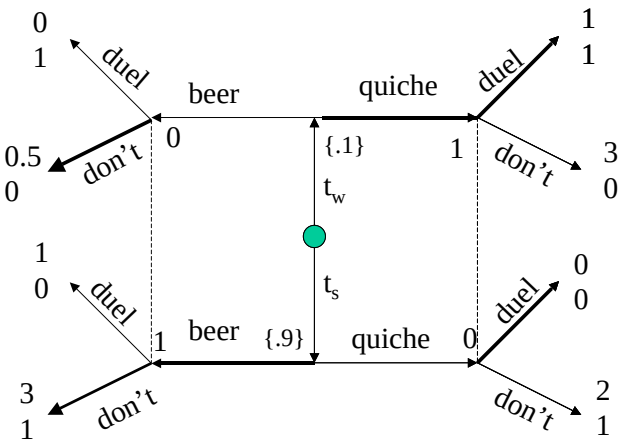



Figure 6:

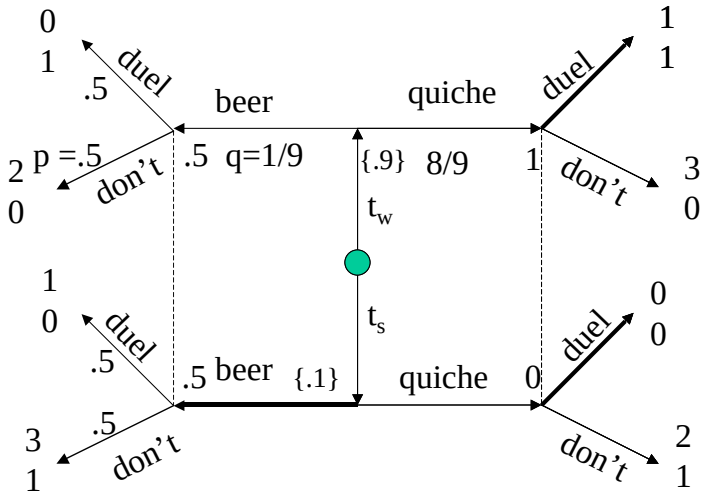



Figure 7:

# 14.12 Game Theory Lecture Notes
# Lecture 20

Muhamet Yildiz

## 1 Adverse Selection

In strategic interactions, a party often knows something that is relevant to the problem but is not known by some other party. In that case, we say that players have asymmetric information. When you are interacting with parties that have private information, if you don't take the necessary precautions, you will likely to end up in a situation that you wouldn't want to be in. This is because the other party will act in his self interest given his information, which will not necessarily be in your interest. This is called adverse selection. For example, the used car that you buy at a low price is likely to be a lemon with many defects, as the owner wouldn't want to sell at such a low price if it were a good car. The candidate who accepted your low wage offer was probably desperately looking for a job and would have accepted even a lower wage. What is worse, he is desperate because he is often fired for misbehavior. The nice looking guy that you have met the other day is likely to be a lousy boyfriend whose relationships do not last long. Why else is he still single? What if he is not single? These kinds of concerns usually make people cautious and prevent trade or other joint decisions that would have been beneficial for all parties and would have been realized if there were no asymmetric information. In this lecture I present examples of adverse selection and illustrate how we compute an equilibrium.

### 1.1 Bargaining

Consider a seller, who owns an object, and a buyer. The value of object is $c \in [0,1]$ for the seller and $v \in [0,1]$ for the buyer. Seller sets a price $p$, and buyer decides whether to

buy it. Seller knows $c$. Let's first assume that the buyer does not know $v$; he thinks that $v$ is uniformly distributed on $[0,1]$. Clearly, buyer will by the object iff $p \leq 1/2$. The seller will set $p = 1/2$ if $c \leq 1/2$, and will set a high price if $c$ is higher. Trade is realized if $c \leq 1/2$, i.e., if it is efficient given what players know. Now, consider the case that buyer knows $v$, and seller thinks that $v$ is uniformly distributed on $[0,1]$. Now, buyer will buy the object if $v \geq p$. Hence the seller's expected payoff is

$$U(p) = \int_p^1 (p-c)\, dv = (p-c)(1-p).$$

Hence, he will set price

$$p = \frac{c+1}{2}.$$

Notice that, when $0 < c \leq 1/2$, the seller sets a higher price. Moreover, when $c < v < (c+1)/2$, they do not trade although the trade would have been mutually beneficial (if they traded at price $(v+c)/2$).

## 1.2 Buying a car

Imagine that you found a job in a suburb, and you need to commute everyday. You want to buy a car. You can buy a car from a private party. Since you need to commute everyday, you think that the value of car for you is higher than the owner. If the value of the car is $v$ for the owner, its value is $v + b$ for you. Here, $v$ is determined by the condition of the car and the maintenance of the car so far. Assume that $v$ is uniformly distributed on $[0,1]$. You don't know $v$, but you know that the owner knows $v$. How much are you willing to pay? If the owner didn't know $v$, you would be willing to pay $b + 1/2$ for the car. (Why?) But the owner knows $v$. He will sell his car at price $p$ only if $p \geq v$. Hence, if you buy the car at $p$, then you know that $v \leq p$. The value of the car for you is then $b + p/2$. (why?) Therefore, you are willing to pay $p$ iff

$$p \leq b + p/2,$$

i.e.,

$$p \leq 2b.$$

If $b < 1/2$, you are willing to pay less when you realize that the seller knows $v$.

How much should you offer, if you were able to make a take it or leave it offer? Your expected payoff from offering a price $p$ is

$$U(p) = \int_0^p (b + v - p)\, dv = bp - p^2/2,$$

which is maximized at

$$p = b.$$

## 1.3 Market for Lemons

Consider a used-car market. There are $n + m$ cars. $n$ of the cars are lemon and require high maintenance. The remaining $m$ cars are peach, i.e., they are great cars that do not require high maintenance. A peach is worth \$2500 to seller, \$3000 to buyer; a lemon is worth \$1000 to seller, \$2000 to buyer. Each seller knows whether his car is a peach or lemon; buyers cannot tell. There are more buyers than cars. We want to compute the market-clearing price. A market-clearing price is a price at which demand and supply curves intersect each other. The supply function is clear. Given any price $p$, the supply is

$$S(p) = \begin{cases} 0 & \text{if } p < 1000 \\ k \leq n \text{ lemons, } 0 \text{ peaches} & \text{if } p = 1000 \\ n \text{ lemons, } 0 \text{ peaches} & \text{if } 1000 < p < 2500 \\ n \text{ lemons, } k \leq m \text{ peaches} & \text{if } p = 2500 \\ n \text{ lemons, } m \text{ peaches} & \text{if } p > 2500. \end{cases}$$

The supply curve is plotted in Figure 1. When a buyer decides to buy a car at this price, he must take into account which cars are sold in the market. Consider first $p > 2500$. At this price, all the cars are in the market. Hence, the probability that a car is peach is

$$\pi = \frac{m}{n + m}.$$

The expected value of a car for a buyer is then

$$3000\pi + (1 - \pi)\, 2000 = 1000\,(\pi + 2)\,.$$

A buyer wants to buy a car iff

$$1000\,(\pi + 2) \geq p. \tag{1}$$

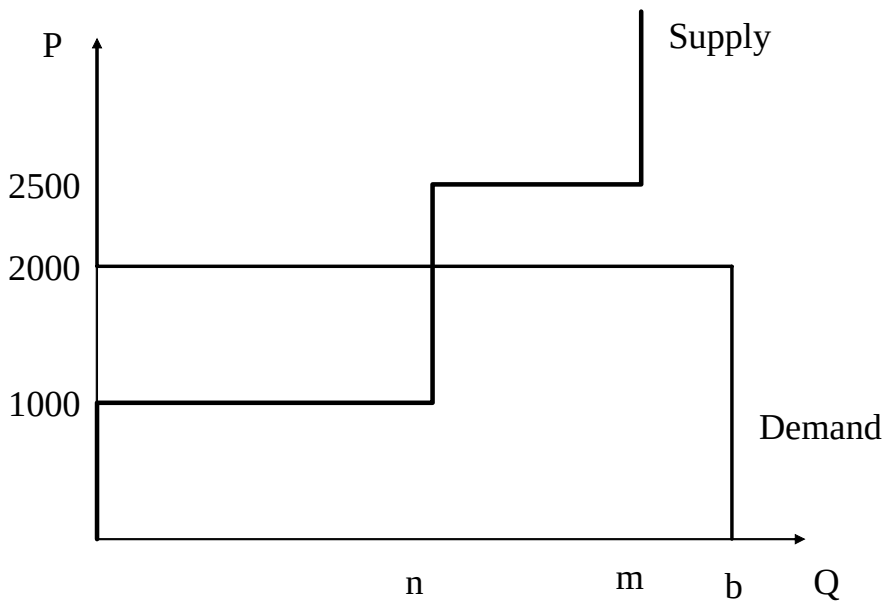



Figure 1:

First consider the case that $\pi < 1/2$, or equivalently $m < n$. That is, there are fewer peaches on the market. In that case, the left-hand side of (1) is less than 2500. Since the right-hand side is greater than 2500, the inequality is never satisfied. Therefore, for any price $p > 2500$, the demand is zero. Similarly, demand remain zero at $p = 2500$, as the probability of peach does not increase as we decrease the price to 2500. At any price $p \in [1000, 2500)$, all the cars in the market are lemon, so that the probability of peach is $\pi = 0$, and therefore the expected value of a car for a buyer is 2000. Hence, a buyer wants to buy a car if $p \leq 2000$. The demand function is given by

$$D(p) = \begin{cases} 0 & \text{if } p > 2000 \\ k \leq b & \text{if } p = 2000 \\ b & \text{if } p < 2000. \end{cases}$$

The market-clearing price is then

$$p^* = 2000,$$

as it is plotted in Figure 1. That is, in competitive equilibrium, only the lemons are traded, and peaches are driven out of market.

Now consider the case $n < m$, i.e., the case that there are more peaches than lemons. In that case, if $p > 2500$, the probability that a car in the market is a peach is $\pi =$

$m/(n+m) > 1/2$. Hence, by (1), a buyer wants to buy a car if

$$p \leq \bar{p} = 1000(\pi + 2),$$

where $\bar{p} > 2500$. At $p = 2500$, $\pi$ depends on the ratio of the peaches that are on the market; recall that a peach owner is indifferent towards selling his car at this price. The demand at this price depend on $\pi \in [0, m/(n+m)]$. When we choose $\pi = 1/2$, the demand at $p = 2500$ can be any number that is less than $b$. When $p < 2000$, the demand is as in the previous case. Hence the demand function is given by

$$D(p) = \begin{cases} 0 & \text{if } p > \bar{p} \\ k \leq b & \text{if } p = \bar{p} \\ b & \text{if } 2500 < p < \bar{p} \\ k \leq b & \text{if } p = 2500 \\ 0 & \text{if } 2000 < p < 2500 \\ k \leq b & \text{if } p = 2000 \\ b & \text{if } p < 2000. \end{cases}$$

This demand function is plotted in Figure 2. Clearly, the demand and supply curves intersect each other at three prices: 2000, 2500, and $\bar{p}$. As in the previous case, at price $p = 2000$, we have only lemons in the market. At price $p = 2500$, all the lemons are sold in the market; only $n$ out of $m$ peaches are in the market so that $\pi = 1/2$, and $n + m$ buyers want to buy a car. Recall that at this price (and given $\pi = 1/2$), the buyers are indifferent. At price $\bar{p}$, all the cars are supplied in the market, and only $n + m$ buyers want to buy a car. The buyers are again indifferent between buying and not buying a car.

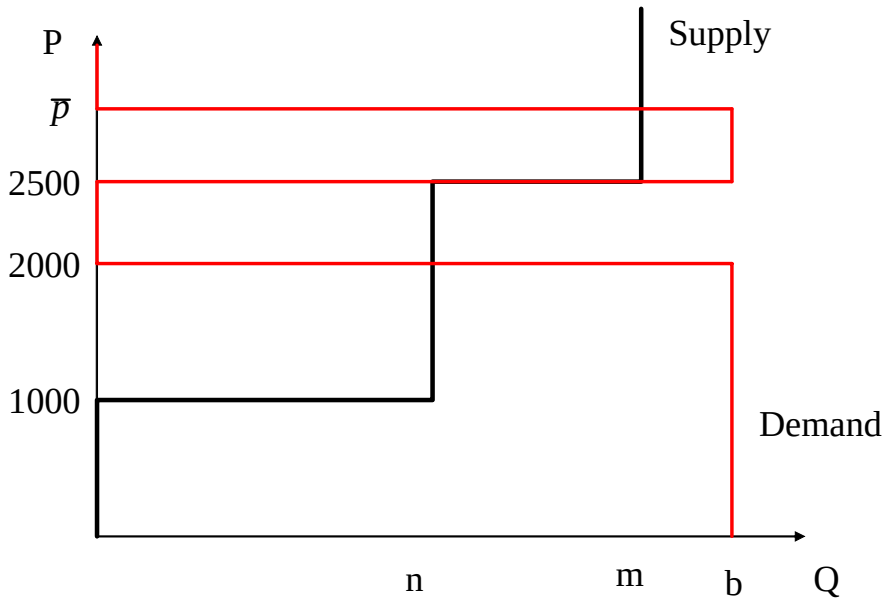



Figure 2: